Selected Proposals

Eric Shen

March 11, 2021

§0 Problems

§0.1 AMC-level

1. (CMC 10B 2020/2) David and Marta are happily married and have three children. In this family of five, the average number of siblings of a member of this family is 4. What is the average of the number of siblings David and Marta have? (Note that David and Marta have siblings who are not part of this family of five.)

2. (CMC 10B 2020/5) The graph of the equation

$$\frac{x+2}{y-7} = \frac{x-5}{y+3}$$

is a line except for two points (a, b) and (c, d). What is $a+b+c+d$?

3. (CMC 10B 2020/7)[1] A three-digit positive integer $\underline{abc}$ is *wavy* if $a < b > c$. How many three-digit positive integers are wavy?

4. (CMC 12A 2020/4) Let n be a positive integer, and consider the set

$$S = \{1, 4, 9, 16, 25, 36, n\}$$

where all seven of its elements are distinct. We replace every odd element k of S with $k+1$, and we replace every even element k of S with $k-1$. It is given that the median of the resulting set is equal to the median of S. How many possible values of n are there?

5. (CMC 10A 2020/8)[2] A circle of radius r centered at the origin of the coordinate plane intersects the graph of $x^2 + y = x + y^2$ at exactly two points. What is the maximum possible value of r?

6. (CMC 12A 2020/10) For which positive integer n does

$$\sqrt{n + \sqrt{n + \sqrt{n + \cdots}}} = \frac{1000}{n + \frac{1000}{n + \frac{1000}{n + \cdots}}}$$

hold?

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[1] This is a cuter version of CIME II 2019/8, proposed by Mason Fang.

[2] Proposed with Allen Baranov.

7. (CMC 12A 2020/9)[3] Positive integers b_4 and b_6 have the property that there is a unique positive integer b_5 such that
$$1 > \frac{b_4}{4^2} > \frac{b_5}{5^2} > \frac{b_6}{6^2}.$$
If $b_4 + b_6 = 24$, what is the unique value of b_5?

8. (CMC 10B 2020/13)[4] A line with negative slope passes through $(1, 1)$ and has x- and y-intercepts of $(a, 0)$ and $(0, b)$ respectively. If $a + b = 8$, what is the value of ab?

9. (CMC 10B 2020/14) Let $P_1P_2 \cdots P_{2020}$ be a regular 2020-gon, and let $P_i = P_{i+2020}$ for all i. For every $i = 1, 2, \ldots, 2020$, draw line P_iP_{i+2}. Into how many regions do these 2020 lines divide the interior of the polygon?

10. (CMC 12A 2020/15) Let ABC be a triangle with $AB = 30$, $BC = 51$, $CA = 63$. Points P and Q lie on $\overline{BC}$, R lies on $\overline{CA}$, and S lies on $\overline{AB}$ such that $PQRS$ is a parallelogram, and the center of $PQRS$ coincides with the centroid of $\triangle ABC$. What is the area of parallelogram $PQRS$?

11. (CMC 12B 2020/13)[5] Oliver selects two nonnegative integers a and b less than 1000 uniformly and at random, and attempts to compute their sum. However, he completely forgets to carry over; so, for instance, to add 13 and 19, he writes down the units digit, 2, discards the carryover, and writes the sum of the tens digits, 2 (thus his answer is 22).

    Suppose that the positive difference between his answer and the correct answer is D. What is the expected value of D?

12. (CMC 12B 2020/14) Let ABC be a triangle with area 1, and let M, N, P be the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. A point X is chosen uniformly and at random in the interior of $\triangle ABC$. What is the expected value of the area of the convex hull of $\{M, N, P, X\}$? (Note: the convex hull of a finite set S in the plane is the smallest convex polygon containing all the elements of S.)

13. (CMC 12A 2020/16)[6] Over all $N > 0$, what is the maximum possible number of permutations a, b, c, d, e of the integers 2, 3, 5, 7, 11 such that it is possible to put parentheses around the expression
$$a \div b \div c \div d \div e$$
so that it equals N?

14. (CMC 10B 2020/19)[7] Let S be a subset of positive integers with the following properties:
    (i) the numbers 1, 2, 3, 4, 5, 6 are elements of S; and
    (ii) for all distinct elements a and b in S, ab is also in S.

    Let K be the sum of the reciprocals of all the elements of S. What is the minimum possible value of K?

15. (CMC 12A 2020/22)[8] In triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively. If the distance from A to $\overline{EF}$ is 2, the distance from B to $\overline{FD}$ is 3 and the distance from C to $\overline{DE}$ is 6, what is the area of $\triangle DEF$?

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[3] The flavortext is obviously based on USA TST 2020/1.
[4] Proposed with Justin Lee.
[5] Proposed independently with Nathan Xiong.
[6] Proposed with Justin Lee.
[7] Proposed with Justin Lee.
[8] Proposed with Justin Lee.

§0.2 AIME-level

16. (CIME I 2020/1) A knight begins on the point $(0, 0)$ in the coordinate plane. From any point (x, y) the knight moves to either $(x + 2, y + 1)$ or $(x + 1, y + 2)$. Find the number of ways the knight can reach the point $(15, 15)$.

17. (CIME II 2020/1) Let ABC be a triangle. The bisector of $\angle ABC$ intersects $\overline{AC}$ at E, and the bisector of $\angle ACB$ intersects $\overline{AB}$ at F. If $BF = 1$, $CE = 2$, and $BC = 3$, then the perimeter of $\triangle ABC$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

18. (CIME II 2020/2) Find the number of nonempty subsets S of $\{1, 2, 3, \dots, 10\}$ such that S has an even number of elements, and the product of the elements of S is even.

19. (Mock AIME 2018/1) The ASCII value of a digit is 48 more than the digit. For instance, the digit 0 has a value of 48, while the digit 7 has a value of 55. Let $g(n)$ be defined as the sum of the ASCII values of the digits of n for all positive integers n, when expressed in base 10. For instance, $g(10) = 97$ and $g(1234) = 202$. The sum of all positive integers n such that $g(n) = n$ is N. Find the remainder when N is divided by 1000.

20. (CIME II 2019/1)[9] Consider the sequence $\{F_n\}$ defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. Suppose that $\mathcal{F} = \{F_2, F_3, \dots, F_{2019}\}$. Then, there are T non-congruent triangles with positive area whose side lengths are all (not necessarily distinct) elements of $\mathcal{F}$. Find the remainder when T is divided by 1000.

21. (Mock AIME 2018/2) There exist two non-intersecting circles with radii 4 and 3. Suppose the length of their common internal tangent is 21. Then, the length of their common external tangent is $\sqrt{d}$, where d is a positive integer. Find d.

22. (Mock AIME 2019/2) There are positive real numbers x, y, z such that $\log_x(yz) = 59$ and $\log_y(zx) = 89$. Find $\log_{xy}(z)$.

23. (Mock AIME 2019 Tiebreaker/Q) Suppose that x and y are real numbers such that
$$\begin{aligned}\log_3(x + y^4) &= \log_3(x - y) + \log_3(x + y), \text{ and}\\ 10 &= \log_3(x - 2y) + \log_3(x + 2y).\end{aligned}$$
Find x.

24. (CMC ARML 2020 I5) Let ABC be a triangle and let M be the midpoint of $\overline{BC}$. The lengths AB, AM, AC form a geometric sequence in that order. The side lengths of $\triangle ABC$ are 2020, 2021, x in some order. Compute the sum of all possible values of x.

25. (Mock AIME 2018/3) Twelve points are chosen uniformly and at random on the circumference of a circle. The probability there exists a diameter $\overline{MN}$ such that all twelve points lie on the same side of $\overline{MN}$ is $\frac{p}{q}$ for relatively prime integers p and q. Find $p + q$.

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[9] Based off a meme problem I've proposed to multiple places, much to their chagrin: if $\triangle(a, b, c)$ denotes the area of the triangle with sides of length a, b, c, compute
$$\sum_{i=1}^{200} \triangle(F_i, F_{i+1}, F_{i+2}).$$

26. (Mock AIME 2018/6) In triangle ABC, X lies on $\overline{AB}$ and Y lies on $\overline{AC}$ such that $\overline{BY}$ bisects $\angle ABC$ and $\overline{CX}$ bisects $\angle ACB$. Segments BY and CX intersect at a point P. Suppose that P lies on the circumcircle of triangle $\triangle AXY$. If $AX = 15$ and $AY = 24$, find AP^2.

27. (Mock AIME 2019/5) Points A and B are randomly and uniformly chosen on the circumference of the circle $x^2 + y^2 = 1$. Find the expected number of ordered pairs of real numbers (p, q) such that the point (p, q) lies on line AB and there exists an integer $1 \leq k \leq 45$ such that $p^2 + q^2 = \sin^2(k^\circ)$.

28. (CIME II 2020/6) An infinite number of buckets, labeled 1, 2, 3, …, lie in a line. A red ball, a green ball, and a blue ball are each tossed into a bucket, such that for each ball, the probability the ball lands in bucket k is 2^{-k}. Given that all three balls land in the same bucket B and that B is even, then the expected value of B can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

29. (CMC ARML 2019 I6)[10] Let $\triangle ABC$ be an isosceles triangle with $AB = AC = 1$, and let E and F be the feet of the altitudes from B and C to sides AC and AB, respectively. If line EF is tangent to the incircle of $\triangle ABC$, compute the perimeter of $\triangle ABC$.

30. (Mock AIME 2019/6) In cyclic quadrilateral $ABCD$, $AB = 8$, $BC = 1$, $CD = 4$, and $DA = 7$. If M denotes the midpoint of $\overline{AC}$, X the intersection of lines AD and BC, Y the intersection of lines AB and CD, and N the midpoint of $\overline{XY}$, then MN can be expressed in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime. Find $p + q + r$.

31. (NEMO 2020 I20) Let ABC be an equilateral triangle and P a point in its interior obeying $AP = 5$, $BP = 7$, $CP = 8$. Line CP intersects $\overline{AB}$ at Q. Compute AQ.

32. (CIME II 2020/8)[11] A committee has an oligarchy, consisting of $A\%$ of the members of the committee. Suppose that $B\%$ of the work is done by the oligarchy. If the average amount of work done by a member of the oligarchy is 16 times the amount of work done by a nonmember of the oligarchy, find the maximum possible value of $B - A$.

33. (CIME II 2020/9)[12] Let $f(x) = x^2 - 2$. There are N real numbers x such that

$$\underbrace{f(f(\dots f}_{\text{2019 times}}(x)\dots)) = \underbrace{f(f(\dots f}_{\text{2020 times}}(x)\dots)).$$

Find the remainder when N is divided by 1000.

34. (Mock AIME 2019 Tiebreaker/V) Call a positive integer *palatable* if when expressed in binary, each contiguous block of zeros that is not a subsequence of another contiguous block of zeros has even length, and each contiguous block of ones that is not a subsequence of another contiguous block of ones has odd length. For example, $57 = 111001_2$ is palatable while $69 = 1000101_2$ is not. Find the number of palatable positive integers N such that $2^{18} < N < 2^{19}$.

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[10] Based on JMO 2019/4.

[11] Proposed with Justin Lee.

[12] Based on CAMO 2020/5, which I also wrote.

35. (CIME II 2019/5) Let $a = 5 + 2i$ and $b = 18 + 13i$, where $i = \sqrt{-1}$. Suppose that z and ω are complex numbers such that
$$\left(z + \frac{1}{z}\right) + \left(\omega + \frac{1}{\omega}\right) = \left(a + \frac{1}{a}\right) \times \left(b + \frac{1}{b}\right), \text{ and}$$
$$\left(z + \frac{1}{z}\right) \times \left(\omega + \frac{1}{\omega}\right) = \left(a^2 + \frac{1}{a^2}\right) + \left(b^2 + \frac{1}{b^2}\right).$$
Then, the largest possible value of $|z + \omega|$ can be expressed as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

36. (Mock AIME 2018/12) In triangle ABC, $AB = 13$, $BC = 14$, $CA = 15$, and a point P lies on $\overline{BC}$. Let Q be the foot of the perpendicular from P to $\overline{AB}$ and R be the foot of the perpendicular from P to $\overline{AC}$. Suppose I_B and I_C are the incenters of triangles $\triangle PBQ$ and $\triangle PCR$, respectively. Then the maximum possible area of $\triangle PI_BI_C$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

37. (Mock AIME 2019/10) Suppose that circles Ω_1 and Ω_2 intersect at P and Q, and that line AB is tangent to Ω_1 and Ω_2 at A and B, respectively, such that Q is closer to $\overline{AB}$ than P. If $AB = 2$, $PA = 20$, and $PB = 19$, then $QA \cdot QB$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

38. (Mock AIME 2019/11) Adam, Bob, and Charlie each flip a coin every day, starting from Day 1, until all three of them have flipped heads at least once. The last of them to flip heads for the first time does so on Day X. The probability that X is even can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

39. (CIME II 2019/12)[13] Max is playing a video game with 99 levels, labeled $1, 2, \ldots, 99$. Whenever Max completes a level, he begins the next one immediately. However, for all $1 \leq n \leq 99$, Max fails the n^{th} level with probability $(n+1)^{-2}$. Whenever he fails a level, he quits for the day and attempts the level again the next day. If Max first attempts the first level on Day 1 and completes the 99^{th} level on Day K, then the expected value of K can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

40. (CIME I 2020/11) An *excircle* of a triangle is a circle tangent to one of the sides of the triangle and the extensions of the other two sides. Let ABC be a triangle with $\angle ACB = 90^\circ$ and let r_A, r_B, r_C denote the radii of the excircles opposite to A, B, C, respectively. If $r_A = 9$ and $r_B = 11$, then r_C can be expressed in the form $m + \sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

41. (CIME II 2019/11) In triangle ABC with incenter I, $AB = 4$, $BC = 5$, and $CA = 6$. If lines AI and BI meet the circumcircle of $\triangle ABC$ again at S and L, respectively, and $\overline{LB}$ and $\overline{LS}$ intersect $\overline{AC}$ at D and E, respectively, then the square of the area of quadrilateral $SIDE$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

42. (Mock AIME 2018/15) Let a, b, c, and d be positive real numbers such that
$$195 = a^2 + b^2 = c^2 + d^2 = \frac{13\left(ac + bd\right)^2}{13b^2 - 10bc + 13c^2} = \frac{5\left(ad + bc\right)^2}{5a^2 - 8ac + 5c^2}.$$

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[13] Proposed with Kyle Lee and Sean Li.

Find the greatest integer that does not exceed $a+b+c+d$.

43. (CIME II 2021/14) Among all pairs of positive integers (x, y) such that $x+y$ divides 2020, randomly and uniformly select one of them. The expected value of $\gcd(x, y)$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m+n$ is divided by 1000.

44. (Mock AIME 2019 Tiebreaker/X)[14] In triangle ABC, $AB = 26$, $BC = 42$, and $CA = 40$. Let ω be the incircle of $\triangle ABC$, and let ω_A be the circle tangent to segment BC and the extensions of lines AB and AC past B and C, respectively. Suppose that ω and ω_A are tangent to $\overline{BC}$ at P and Q, respectively, and that X and Y lie on ω and ω_A, respectively, such that $\angle AXP = \angle AYQ = 90^\circ$. If M is the midpoint of $\overline{BC}$ and Z is the intersection of $\overline{PX}$ and $\overline{QY}$, find MZ^2.

45. (Mock AIME 2019 Tiebreaker/Z) In triangle ABC, $AB = 14$, $BC = 17$, and $CA = 15$. The incircle of $\triangle ABC$ touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively; and $\overline{AD}$ meets the incircle of $\triangle ABC$ again at T. Suppose that points M and U lie on $\overline{AC}$ and points N and V lie on $\overline{AB}$ such that $\overline{MN}$ is tangent to the incircle of $\triangle ABC$ at T, and $\overline{EV}$ and $\overline{FU}$ intersect at T. Then, there exist relatively prime positive integers p and q such that $\frac{MU}{NV} = \frac{p}{q}$. Find $p+q$.

46. (Mock AIME 2019/9) For relatively prime positive integers a and b and positive real numbers c and θ, let K denote the area of the triangle with sides of length $a \sin\theta$, $b \cos\theta$, and $c \tan\theta$, given that it is positive. Suppose that if a and b remain fixed, and c and θ vary, then K achieves a maximum when $c = 85$. Find the sum of all distinct possible values of $a+b$.

47. (CIME I 2020/14) Let ABC be a triangle with sides $AB = 5$, $BC = 7$, $CA = 8$. Denote by O and I the circumcenter and incenter of $\triangle ABC$, respectively. The incircle of $\triangle ABC$ touches $\overline{BC}$ at D, and line OD intersects the circumcircle of $\triangle AID$ again at K. Then the length of DK can be expressed in the form $\frac{m\sqrt{n}}{p}$, where m, n, p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime. Find $m+n+p$.

48. (CIME I 2021/14)[15] Let ABC be an acute triangle with circumcenter O. Points X and Y lie on the tangent to the circumcircle of $\triangle ABC$ at A so that $\overline{BX} \perp \overline{AC}$ and $\overline{CY} \perp \overline{AB}$. Line AO intersects $\overline{BC}$ at D. Suppose that $AO = 25$, $BC = 49$, and $\overline{BY} \parallel \overline{CX}$. Compute AD.

49. (Mock AIME 2019/13) Circles Γ_1, Γ_2, and Γ_3 are pairwise externally tangent and have diameters of length 70, 99, and 55, respectively. Suppose that Γ_2 and Γ_3 touch at X, Γ_3 and Γ_1 touch at Y, and Γ_1 and Γ_2 touch at Z. A point A is chosen on the minor arc YZ of Γ_1. Ray AZ intersects Γ_2 again at B, ray BX intersects Γ_3 again at C, ray CY intersects Γ_1 again at D, ray DZ intersects Γ_2 again at E, ray EX intersects Γ_3 again at F, and ray FY intersects Γ_1 again at G. Find the maximum possible value of $AB + BC + CD + DE + EF + FG$.

50. (CIME II 2020/15) Let $P_1P_2 \cdots P_{72}$ be a regular dodecagon with area 1, and let $P_i = P_{i+72}$ for all integers i. Let S be the sum of the squares all positive integers $a < 72$ such that
    - for all i, $P_{i-3a} \neq P_{i+a}$ and $P_{i-a} \neq P_{i+3a}$;

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[14] Based on USA TST 2015/1.

[15] Proposed with Tovi Wen.

- for all i, lines $P_{i-3a}P_{i+a}$ and $P_{i-a}P_{i+3a}$ are not parallel, do not coincide, and intersect at a point Q_i; and
- the points Q_1, Q_2, ..., Q_{72} form a polygon with positive, rational area.

Find the remainder when S is divided by 1000.

51. (Mock AIME 2018/14)[16] Suppose 2019 chicks are sitting in a circle. Suddenly, each chick randomly pecks either the chick on its left or the chick on its right with equal probability. Let k be the number of chicks that were not pecked. The probability k is odd can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p+q$ is divided by 1000.

52. (Mock AIME 2019/15) Let ABC be a triangle with $AB = 11$, $BC = 19$, $CA = 20$. Let O denote its circumcenter, and let D, E, F be the feet of the altitudes from A, B, C, respectively. Points X, Y are the feet of the altitudes from E, F onto $\overline{AD}$. If $\overline{AO}$, $\overline{EF}$ intersect at Z, then there is a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. Let $\overline{ZT}$, $\overline{AD}$ intersect at P. Given that there exist relatively prime positive integers m, n such that $\frac{PX}{PY} = \frac{m}{n}$, find $m+n$.

53. (OMO Spring 2020/28) Let A_0BC_0D be a convex quadrilateral inscribed in a circle ω. For all integers $i \geq 0$, let P_i be the intersection of $\overline{A_iB}$ and $\overline{C_iD}$, let Q_i be the intersection of $\overline{A_iD}$ and $\overline{BC_i}$, let M_i be the midpoint of $\overline{P_iQ_i}$, and let $\overline{M_iA_i}$ and $\overline{M_iC_i}$ intersect ω again at A_{i+1} and C_{i+1}, respectively. The circumcircles of $\triangle A_3M_3C_3$ and $\triangle A_4M_4C_4$ intersect at two points U and V.

    If $A_0B = 3$, $BC_0 = 4$, $C_0D = 6$, $DA_0 = 7$, then UV can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c such that $\gcd(a, c) = 1$ and b is squarefree. Find $100a + 10b + c$.

§0.3 Olympiad-level

54. (CJMO 2020/1) Let N be a positive integer, and let S be the set of all tuples with positive integer elements and a sum of N. For instance, $t_1 = (N)$, $t_2 = (1, 1, N-2)$, $t_3 = (1, N-1)$, and $t_4 = (N-1, 1)$ are all distinct tuples in S. For all tuples t, let $p(t)$ denote the product of all the elements of t. For instance, $p(t_1) = N$, $p(t_2) = N-2$, and $p(t_3) = p(t_4) = N-1$.

    Evaluate the expression (where we sum over all elements t of S)

    $$\sum_{t \in S} p(t).$$

55. (CJMO 2021/1) Let ABC be an acute triangle, and let the feet of the altitudes from A, B, C to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ be D, E, F, respectively. Points X and Y lie on lines CF and BE respectively such that $\angle XAD = \angle DAB$ and $\angle YAD = \angle DAC$. Prove that X, D, Y are collinear.

56. (CJMO 2019/3) Let I be the incenter of $\triangle ABC$, and M be the midpoint of $\overline{BC}$. Let Ω be the nine-point circle of $\triangle BIC$. Suppose that $\overline{BC}$ intersects Ω at a point $D \neq M$. If Y is the intersection of $\overline{BC}$ and the A-intouch chord, and X is the projection of Y onto

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[16] Based on Luke Robitaille's winning countdown question at Mathcounts Nationals 2019:

> In a barn, 100 chicks sit peacefully in a circle. Suddenly, each chick randomly pecks the chick immediately to its left or right. What is the expected number of un-pecked chicks?

$\overline{AM}$, prove that X lies on Ω, and the intersection of the tangents to Ω at D and X lies on the A-intouch chord of $\triangle ABC$.

(The nine-point circle of $\triangle ABC$ is the circumcircle of its medial triangle, and if the incircle touches $\overline{AC}$ and $\overline{AB}$ at E and F, respectively, then $\overline{EF}$ is the A-intouch chord.)

57. (CAMO 2020/1)[17] Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning f takes positive real numbers to positive real numbers) be a nonconstant function such that for any positive real numbers x and y,

$$f(x)f(y)f(x+y) = f(x) + f(y) - f(x+y).$$

Prove that there is a constant $a > 1$ such that

$$f(x) = \frac{a^x - 1}{a^x + 1}$$

for all positive real numbers x.

58. (c851151h1975686p13709352)[18] Prove that for every integer k, there is a permutation $(a_1, a_2, \dots, a_{2010})$ of the numbers $(1, 2, \dots, 2010)$ such that

$$a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_{2009}a_{2010} + a_{2010}a_1 \equiv k \pmod{2011}.$$

59. (CJMO 2020/5) Let ABC be a triangle, and D be a point on the internal angle bisector of $\angle BAC$ but not on the circumcircle of $\triangle ABC$. Suppose that the circumcircle of $\triangle ABD$ intersects $\overline{AC}$ again at P and the circumcircle of $\triangle ACD$ intersects $\overline{AB}$ again at Q. Denote by O_1 and O_2 the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Prove that the circumcenters of $\triangle ABC$, $\triangle APQ$, and $\triangle AO_1O_2$ are collinear.

60. (CJMO 2021/5) Let $ABCD$ be a cyclic quadrilateral, and let M and N be the midpoints of $\overline{AC}$ and $\overline{BD}$. The circumcircles of $\triangle ABM$ and $\triangle CDM$ intersect again at P; the circumcircles of $\triangle ABN$ and $\triangle CDN$ intersect again at Q; the circumcircles of $\triangle ADM$ and $\triangle BCM$ intersect again at R; and the circumcircles of $\triangle ADN$ and $\triangle BCN$ intersect again at S. Prove that $\overline{PQ} \parallel \overline{RS}$.

61. (CAMO 2021/4) Let ABC be a triangle and let M be the midpoint of $\overline{BC}$. The circumcircle of $\triangle ABM$ intersects $\overline{AC}$ again at P, and the circumcircle of $\triangle ACM$ intersects $\overline{AB}$ again at Q. Select point T on the circumcircle of $\triangle MPQ$ such that $\overline{MT} \parallel \overline{PQ}$, and let ω be the circle through T tangent to $\overline{BC}$ at M. The circumcircles of $\triangle ABM$ and $\triangle ACM$ intersect ω again at X and Y. Prove that line XY is tangent to the circumcircle of $\triangle ABC$.

62. (CAMO 2020/4) Let ABC be a triangle and Q a point on its circumcircle. Let E and F be the reflections of Q over $\overline{AB}$ and $\overline{AC}$, respectively. Select points X and Y on line EF such that $\overline{BX} \parallel \overline{AC}$ and $\overline{CY} \parallel \overline{AB}$, and let M and N be the reflections of X and Y over B and C respectively. Prove that M, Q, N are collinear.

63. (CAMO 2020/3) Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Point M is the midpoint of $\overline{EF}$, and T is the point on ω such that $\overline{DT}$ is a diameter. Line MT meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines DF and DE intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.

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[17] Proposed with Raymond Feng.

[18] Proposed with Raymond Feng. We underestimated how well-received this question was and placed it on a practice Mock JMO for our students.

64. (CAMO 2020/5) Let $f(x) = x^2 - 2$. Prove that for all positive integers n, the polynomial

$$P(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)\dots)) - x$$

can be factored into two polynomials with integer coefficients and equal degree.

65. (CAMO 2021/5)[19] Do there exist positive integers m and n such that

$$\frac{n^2 - 1}{m^2 - n^2 - 1}$$

is also a positive integer?

66. (CAMO 2021/3) Let ABC be an scalene triangle with circumcircle Γ and orthocenter H, and let K and M be the midpoints of $\overline{AH}$ and $\overline{BC}$, respectively. Line AH intersects Γ again at T, and line KM intersects Γ at U and V. Lines TU and TV intersect lines AB and AC at X and Y, respectively, and point W lies on line KM such that $\overline{AW} \perp \overline{HM}$. If Z is the reflection of A over W, prove that X, Y, Z are collinear.

67. (CAMO 2020/6)[20] Let n be a positive integer. Eric and a squid play a turn-based game on an infinite grid of unit squares. Eric's goal is to capture the squid by moving onto the same square as it.

    Initially, all the squares are colored white. The squid begins on an arbitrary square in the grid, and Eric chooses a different square to start on. On the squid's turn, it performs the following action exactly 2020 times: it chooses an adjacent unit square that is white, moves onto it, and sprays the previous unit square either black or gray. Once the squid has performed this action 2020 times, all squares colored gray are automatically colored white again, and the squid's turn ends. Moreover, the squid is claustrophobic, so at no point in time is it ever surrounded by a closed loop of black or gray squares. On Eric's turn, he performs the following action at most n times: he chooses an adjacent unit square that is white and moves onto it. Note that the squid can trap Eric in a closed region, so that Eric can never win.

    Eric wins if he ever occupies the same square as the squid. Suppose the squid has the first turn, and both Eric and the squid play optimally. Both Eric and the squid always know each other's location and the colors of all the squares. Find all positive integers n such that Eric can win in finitely many moves.

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[19] Proposed with Sean Li. Based on China TST 2018/3/6.

[20] Proposed with Raymond Feng in an attempt to imitate Nikolai Beluhov. The wording is kind of terrible and I have yet to reword it; the Chinese version—found at `http://cmc.ericshen.net/zh/CMC-2020/zh-CAMO-2020.pdf`—is much easier to read.

§1 CMC 10B 2020/2

David and Marta are happily married and have three children. In this family of five, the average number of siblings of a member of this family is 4. What is the average of the number of siblings David and Marta have? (Note that David and Marta have siblings who are not part of this family of five.)

Answer. 7

Let David, Marta have a, b siblings. The children each have 2 sublings, so

$$\frac{a+b+2+2+2}{5}=4 \implies \frac{a+b}{2}=7.$$

§2 CMC 10B 2020/5

The graph of the equation

$$\frac{x+2}{y-7}=\frac{x-5}{y+3}$$

is a line except for two points (a,b) and (c,d). What is $a+b+c+d$?

Answer. 7

They're just $(-2,7)$, $(5,-3)$. The answer is 7.

§3 CMC 10B 2020/7

A three-digit positive integer $\underline{abc}$ is *wavy* if $a<b>c$. How many three-digit positive integers are wavy?

Answer. 240

By casework on b, the answer is $1\cdot 2+2\cdot 3+\cdots+8\cdot 9=240$.

§4 CMC 12A 2020/4

Let n be a positive integer, and consider the set

$$S=\{1,4,9,16,25,36,n\}$$

where all seven of its elements are distinct. We replace every odd element k of S with $k+1$, and we replace every even element k of S with $k-1$. It is given that the median of the resulting set is equal to the median of S. How many possible values of n are there?

Answer. 2

The resulting set is $S'=\{2,3,10,15,26,35,n\pm 1\}$. If the median of S is not n, then $n\pm 1$ must be the median of S, and also one of $\{1,4,9,16,25,36\}$. This two assertions contradict each other, so n must be the median of S.

This forces $9<n<16$. If the median of S' is also $n\pm 1$, then $n\neq n\pm 1$, so the median of S' is either 9 or 16. It follows that $n\in\{10,15\}$, both of which can be easily seen to work. The answer is 2.

§5 CMC 10A 2020/8

A circle of radius r centered at the origin of the coordinate plane intersects the graph of $x^2+y=x+y^2$ at exactly two points. What is the maximum possible value of r?

Answer. $1/\sqrt{2}$

The given curve rearranges to $(x-y)(x+y-1)=0$, so it is the union of the lines $y=x$ and $y=1-x$. Each circle intersects $y=x$ twice, and it intersects twice $y=1-x$ when $r>1/\sqrt{2}$. We can check $r=1/\sqrt{2}$ works.

§6 CMC 12A 2020/10

For which positive integer n does

$$\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}=\frac{1000}{n+\frac{1000}{n+\frac{1000}{n+\cdots}}}$$

hold?

Answer. 90

Let the common value be x. Write

$$x=\sqrt{n+x}=\frac{1000}{n+x}.$$

If $y=n+x$, then $\sqrt{y}=1000/y$, so $y^{3/2}=1000$ and $y=100$. This tells us the common value is $x=10$, so $n=100-10=90$.

§7 CMC 12A 2020/9

Positive integers b_4 and b_6 have the property that there is a unique positive integer b_5 such that

$$1>\frac{b_4}{4^2}>\frac{b_5}{5^2}>\frac{b_6}{6^2}.$$

If $b_4+b_6=24$, what is the unique value of b_5?

Answer. 12

To pick a starting point for our search, we consider the least b_4 so that $\frac{b_4}{4^2}>\frac{b_6}{6^2}$; that is,

$$\frac{9}{4}>\frac{b_6}{b_4}=\frac{24-b_4}{b_4}\implies b_4>\frac{96}{13}>7.$$

As such, we check $(b_4,b_6)=(8,16)$. This yields

$$\frac{8}{4^2}>\frac{12}{5^2}>\frac{16}{6^2},$$

so $b_5=12$.

> **Remark.** To see that no other $b_4 > 8$ work, check that in these cases,
>
> $$\frac{b_4}{4^2} - \frac{b_6}{6^2} \geq \frac{9}{16} - \frac{15}{36} = \frac{7}{48} > \frac{2}{25},$$
>
> so we conclude by the Pidgeonhole Principle that there are at least two $b_5/5^2$, with $b_5 \in \mathbb{Z}$, obeying
>
> $$\frac{b_4}{4^2} > \frac{b_5}{5^2} > \frac{b_6}{6^2}.$$

§8 CMC 10B 2020/13

A line with negative slope passes through $(1,1)$ and has x- and y-intercepts of $(a,0)$ and $(0,b)$ respectively. If $a+b=8$, what is the value of ab?

Answer. 8

The equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

thus we have

$$1 = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab},$$

and the answer is 8.

§9 CMC 10B 2020/14

Let $P_1P_2 \cdots P_{2020}$ be a regular 2020-gon, and let $P_i = P_{i+2020}$ for all i. For every $i = 1, 2, \ldots, 2020$, draw line P_iP_{i+2}. Into how many regions do these 2020 lines divide the interior of the polygon?

Answer. 4041

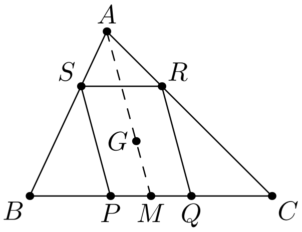
For general $n = 2020$, the answer is $2n + 1 = 4041$. Basically the lines divide the n-gon into a "middle" n-gon, and each vertex and side determines another region.

§10 CMC 12A 2020/15

Let ABC be a triangle with $AB = 30$, $BC = 51$, $CA = 63$. Points P and Q lie on $\overline{BC}$, R lies on $\overline{CA}$, and S lies on $\overline{AB}$ such that $PQRS$ is a parallelogram, and the center of $PQRS$ coincides with the centroid of $\triangle ABC$. What is the area of parallelogram $PQRS$?

Answer. 336

Let G be the centroid and M the midpoint of $\overline{BC}$. In general, the area of $PQRS$ is $\frac{4}{9}$ of the area of $\triangle ABC$.



Let $d(X, \ell)$ be the distance from X to ℓ and $d(\ell_1, \ell_2)$ be the distance between parallel lines ℓ_1, ℓ_2. Since $d(A, \overline{BC}) = 3d(G, \overline{BC})$ and G is the center of $PQRS$, we have $d(\overline{RS}, \overline{BC}) = \frac{2}{3}d(A, \overline{BC})$. This implies $AR = \frac{1}{3}AC$ and $AS = \frac{1}{2}AB$.

Now $\overline{AG}$ bisects $\overline{RS}$ by homothety, so the midpoint of $\overline{PQ}$ is M. Then $PQ = RS = \frac{1}{3}BC$, so $BP = PQ = QC$. Finally

$$\frac{[ASR]}{[ABC]} = \frac{1}{9}, \quad \frac{[BPS]}{[ABC]} = \frac{2}{9}, \quad \frac{[CQR]}{[ABC]} = \frac{2}{9},$$

and combining these results yields the desired conclusion. Alternatively the base of $PQRS$ is $\frac{1}{3}$ that of $\triangle ABC$, and its height is $\frac{2}{3}$ that of $\triangle ABC$, and the result readily follows as well.

From the given numbers, $[ABC] = 756$, so the answer is 336.

§11 CMC 12B 2020/13

Oliver selects two nonnegative integers a and b less than 1000 uniformly and at random, and attempts to compute their sum. However, he completely forgets to carry over; so, for instance, to add 13 and 19, he writes down the units digit, 2, discards the carryover, and writes the sum of the tens digits, 2 (thus his answer is 22).

Suppose that the positive difference between his answer and the correct answer is D. What is the expected value of D?

Answer. 999/2

Linearity of expectation; see `https://aops.com/community/c1035147h2000681p13982910`.

§12 CMC 12B 2020/14

Let ABC be a triangle with area 1, and let M, N, P be the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. A point X is chosen uniformly and at random in the interior of $\triangle ABC$. What is the expected value of the area of the convex hull of $\{M, N, P, X\}$? (Note: the convex hull of a finite set S in the plane is the smallest convex polygon containing all the elements of S.)

Answer. $\frac{5}{16}$

There is a $\frac{1}{4}$ change X is in $\triangle MNP$, and the area is $\frac{1}{4}$. Otherwise, without loss of generality X is in $\triangle ANP$. The expected area of $\triangle XNP$ is $\frac{1}{3}$ that of $\triangle ANP$ by symmetry (since the areas of $\triangle XNP$, $\triangle XPA$, $\triangle XAM$ have fixed sum); i.e. $\frac{1}{12}$, so the expected area in this case is $\frac{1}{3}$. The answer is

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} = \frac{5}{16}.$$

§13 CMC 12A 2020/16

Over all $N > 0$, what is the maximum possible number of permutations a, b, c, d, e of the integers 2, 3, 5, 7, 11 such that it is possible to put parentheses around the expression

$$a \div b \div c \div d \div e$$

so that it equals N?

Answer. 36

Note that a and b end up in the numerator and denominator respectively, but c, d, e can go anywhere. That is, for all $x, y, z \in \{1, -1\}$, the expression can attain $ab^{-1}c^x d^y e^z$. Optimally the exponents of 2, 3, 5, 7, 9 in N are all ± 1, whereas all other exponents are 0. Say that there are p exponents of 1 and q exponents of -1.

We can choose a in p ways and b in q ways. Next, c, d, e may be any of the 6 permutations of the remaining primes, and x, y, z follow accordingly. Then the number of permutations is given by $6pq \le 6 \cdot 2 \cdot 3 = 36$, the answer.

§14 CMC 10B 2020/19

Let S be a subset of positive integers with the following properties:

(i) the numbers 1, 2, 3, 4, 5, 6 are elements of S; and
(ii) for all distinct elements a and b in S, ab is also in S.

Let K be the sum of the reciprocals of all the elements of S. What is the minimum possible value of K?

Answer. $\frac{53}{15}$

The minimal set S by inclusion only contains numbers with prime factors within $\{2, 3, 5\}$. Since 4, 6, 10 are in S and we may repeatedly multiply by 2, 3, 5, any even number with prime factors in $\{2, 3, 5\}$ must be an element of S. Furthermore $15 \in S$, and repeatedly multiplying by 3, 5 we have any number with both 3, 5 as prime factors is an element of S.

In conclusion, S contains numbers with prime factors within $\{2, 3, 5\}$, except 3^i, $i \ge 2$ and 5^i, $i \ge 2$. The answer is

$$\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots\right) - \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots\right) - \left(\frac{1}{5^2} + \frac{1}{5^3} + \cdots\right) = \frac{53}{15}.$$

§15 CMC 12A 2020/22

In triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively. If the distance from A to $\overline{EF}$ is 2, the distance from B to $\overline{FD}$ is 3 and the distance from C to $\overline{DE}$ is 6, what is the area of $\triangle DEF$?

Answer. 6

Note that A, B, C are the D-, E-, F-excenters of $\triangle DEF$. Henceforth let $a = EF$, $b = FD$, $c = DE$, $s = \frac{1}{2}(a+b+c)$, and let K be the area. Hence

$$\frac{K}{s-a} = 2, \quad \frac{K}{s-b} = 3, \quad \frac{K}{s-c} = 6.$$

From this,

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{s-a}{K} + \frac{s-b}{K} + \frac{s-c}{K} = \frac{s}{K},$$

so $s = K$.

By Heron's formula, $K^2 = s(s-a)(s-b)(s-c)$, so $K = (s-a)(s-b)(s-c)$ and

$$K^2 = \frac{K}{s-a} \cdot \frac{K}{s-b} \cdot \frac{K}{s-c} = 36,$$

whence $K = 6$.

§16 CIME I 2020/1

A knight begins on the point $(0, 0)$ in the coordinate plane. From any point (x, y) the knight moves to either $(x+2, y+1)$ or $(x+1, y+2)$. Find the number of ways the knight can reach the point $(15, 15)$.

Answer. 252

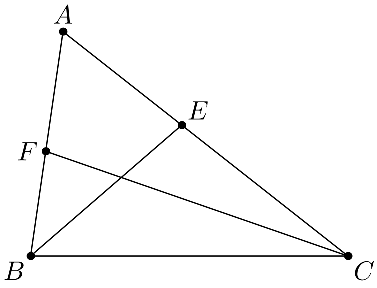
Say there are A moves of the form $(x, y) \mapsto (x+2, y+1)$ and B moevs of the form $(x, y) \mapsto (x+1, y+2)$. The final position is $(15, 15)$, so $2A+B = A+2B = 15$. It follows that $A = B = 5$.

There are 10 moves in total, and 5 are of one type and 5 of the other. All permutations of these 10 moves are valid, so the answer is $\binom{10}{5} = 252$.

§17 CIME II 2020/1

Let ABC be a triangle. The bisector of $\angle ABC$ intersects $\overline{AC}$ at E, and the bisector of $\angle ACB$ intersects $\overline{AB}$ at F. If $BF = 1$, $CE = 2$, and $BC = 3$, then the perimeter of $\triangle ABC$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer. 067



Let $AF = x$ and $AE = y$. By the Angle Bisector theorem

$$x = \frac{y+2}{3} \quad \text{and} \quad \frac{y}{2} = \frac{x+1}{3}.$$

Thus $3x = y + 2$ and $3y = 2x + 2$. Substituting $y = 3x - 2$, we have $9x - 6 = 2x + 2$, so $x = 8/7$ and $y = 10/7$. The desired perimeter is $x + y + 6 = 60/7$, and the requested sum is $60 + 7 = 67$.

§18 CIME II 2020/2

Find the number of nonempty subsets S of $\{1, 2, 3, \ldots, 10\}$ such that S has an even number of elements, and the product of the elements of S is even.

Answer. 496

The number of S with an even number of elements is 2^9, by choosing for $i = 1, \ldots, 9$ whether i is in S, and whether 10 is in S is dependent on parity.

All S without an even product of elements contains only odd elements. The number of such S is 2^4, by choosing for $i = 1, 3, 5, 7$ whether i is in S, and whether 9 is in S is dependent on parity.

The answer is $2^9 - 2^4 = 496$.

§19 Mock AIME 2018/1

The ASCII value of a digit is 48 more than the digit. For instance, the digit 0 has a value of 48, while the digit 7 has a value of 55. Let $g(n)$ be defined as the sum of the ASCII values of the digits of n for all positive integers n, when expressed in base 10. For instance, $g(10) = 97$ and $g(1234) = 202$. The sum of all positive integers n such that $g(n) = n$ is N. Find the remainder when N is divided by 1000.

Answer. 545

First note that a one-digit integer cannot satisfy the condition, as $g(n) \geq 48$. In addition, integers with $k \geq 4$ digits cannot satisfy the condition as well, as $g(n) \leq (48 + 9)k = 57k < n$. Now, we only need to consider two-digit and three-digit integers.

If n has two digits, let $n = \overline{ab}$. Then, $g(n) = (48+a)+(48+b) = 96+a+b$. Also, $n = 10a+b$, so $96 + a + b = 10a + b$, so $9a = 96$. This is clearly not possible.

If n has three digits, let $n = \overline{abc}$. Then, $g(n) = (48+a)+(48+b)+(48+c) = 144+a+b+c$. Also, $n = 100a + 10b + c$. Thus, $144 + a + b + c = 100a + 10b + c$, so $144 = 99a + 9b$, and $9(11a + b) = 144 \implies 11a + b = 16$. The only pair (a, b) that satisfies this is $(1, 5)$. Since c can be any digit, the desired sum is $150 + 151 + \cdots + 159 = 309 \cdot 5 = 1545$, and the requested remainder is 545.

§20 CIME II 2019/1

Consider the sequence $\{F_n\}$ defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. Suppose that $\mathcal{F} = \{F_2, F_3, \ldots, F_{2019}\}$. Then, there are T non-congruent triangles with positive area whose side lengths are all (not necessarily distinct) elements of $\mathcal{F}$. Find the remainder when T is divided by 1000.

Answer. 187

We claim that no scalene triangle works. Assume for the sake of contradiction that a triangle with sides $F_a < F_b < F_c$ works. Then,

$$F_a + F_b \leq F_{b-1} + F_b = F_{b+1} \leq F_c,$$

a contradiction. Hence, the triangle is isosceles. If the leg has length $F_2 = 1$, then only F_2 works for the base. If $i > 2$ and the leg has length F_i, since

$$2F_i > F_{i-1} + F_i = F_{i+1}, \text{ but } 2F_i < F_i + F_{i+1} = F_{i+2},$$

the other leg must be $\{F_2, F_3, \dots, F_{i+1}\}$. However, for $i = 2019$, F_{2020} is not in $\mathcal{F}$, so the requested remainder is

$$\begin{aligned} 1 + 3 + 4 + \cdots + 2017 + 2018 + 2018 &\equiv \frac{2019 \cdot 2020}{2} - 3 \\ &\equiv 19 \cdot 10 - 3 \\ &\equiv 187 \pmod{1000}, \end{aligned}$$

and we are done.

§21 Mock AIME 2018/2

There exist two non-intersecting circles with radii 4 and 3. Suppose the length of their common internal tangent is 21. Then, the length of their common external tangent is $\sqrt{d}$, where d is a positive integer. Find d.

Answer. 489

Suppose the center of the circle with radius 4 is P and the center of the circle with radius 3 is Q. Suppose the common internal tangent touches circle P at A and circle Q at B. Let $\overline{PQ}$ intersect $\overline{AB}$ at X. Then, by AA, $\triangle PAX \sim \triangle QBX$. It follows that $\frac{AX}{BX} = \frac{AP}{BQ} = \frac{4}{3}$. Since $AX + BX = 21$, $AX = 12$ and $BX = 9$. It follows that $PX = 4\sqrt{10}$ and $QX = 3\sqrt{10}$, so $PQ = 7\sqrt{10}$.

Suppose the common external tangent touches circle P at A' and Q at B'. Suppose C lies on $\overline{A'P}$ such that $\overline{A'P} \perp \overline{CQ}$. Then, since $\overline{A'P} \parallel \overline{B'Q}$, $CP = 1$ and $AC = 3$. Note that since $\overline{CQ} \parallel \overline{A'B'}$, $CQ = A'B'$. It is easy to see that $CQ = \sqrt{PQ^2 - CP^2} = \sqrt{489}$, and the answer is 489.

§22 Mock AIME 2019/2

There are positive real numbers x, y, z such that $\log_x(yz) = 59$ and $\log_y(zx) = 89$. Find $\log_{xy}(z)$.

Answer. 035

First solution We are given that $x^{59} = yz$ and $y^{89} = zx$. Then,

$$z = \frac{x^{59}}{y} = \frac{y^{89}}{x},$$

and multiplying gives $x^{60} = y^{90}$, or $x^2 = y^3$. Thus, there exists a real number t such that $x = t^3$ and $y = t^2$. Furthermore,

$$z = \frac{(t^3)^{59}}{t^2} = t^{175},$$

whence

$$\log_{xy}(z) = \frac{\log_t z}{\log_t x + \log_t y} = \frac{175}{3+2} = 35,$$

the answer.

Second solution Let $a = 59$ and $b = 89$. We have that $x^a = yz$ and $y^b = zx$. The latter gives $x = y^b z^{-1}$, so plugging into the first, $y^{ab}z^{-a} = yz$, whence $y^{ab-1} = z^{a+1}$. Then,

$$\log_z y = \frac{a+1}{ab-1} \text{ and } \log_z x = \frac{ab+b}{ab-1} - 1 = \frac{b+1}{ab-1}.$$

It follows that

$$\frac{1}{\log_{xy} z} = \log_z x + \log_z y = \frac{b+1}{ab-1} + \frac{a+1}{ab-1} = \frac{a+b+2}{ab-1} = \frac{1}{35},$$

and the answer is 35.

§23 Mock AIME 2019 Tiebreaker/Q

Suppose that x and y are real numbers such that

$$\log_3(x+y^4) = \log_3(x-y) + \log_3(x+y), \text{ and}$$
$$10 = \log_3(x-2y) + \log_3(x+2y).$$

Find x.

Answer. 245

The first relation gives us $x + y^4 = (x-y)(x+y) = x^2 - y^2$, so $x^2 - x = y^4 + y^2$ and

$$\left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4} = y^4 + y^2 + \frac{1}{4} = \left(y^2 + \frac{1}{2}\right)^2.$$

This implies that either $x = y^2 + 1$ or $x = -y^2$. In the latter case, the second condition gives a contradiction, so $x = y^2 + 1$. The second condition now gives

$$3^{10} = (x-2y)(x+2y) = x^2 - 4y^2 = (y^2+1)^2 - 4y^2 = (y^2-1)^2,$$

so $y^2 - 1 = 243$, and $x = y^2 + 1 = 245$, the answer.

§24 CMC ARML 2020 I5

Let ABC be a triangle and let M be the midpoint of $\overline{BC}$. The lengths AB, AM, AC form a geometric sequence in that order. The side lengths of $\triangle ABC$ are 2020, 2021, x in some order. Compute the sum of all possible values of x.

Answer. $8082 + \sqrt{2}$

The key claim is this:

Claim. $AM^2 = AB \cdot AC$ if and only if $BC = |AB - AC|\sqrt{2}$

Proof. The proof is by the Median formula

$$AM^2 = \frac{2\left(AB^2 + AC^2\right) - BC^2}{4},$$

which is a corollary of Stewart's theorem. The above equation rearranges to

$$\frac{BC^2}{2} = AB^2 + AC^2 - 2AM^2,$$

and the claim follows. $\square$

Evidently all triangle satisfying $BC = |AB - AC|\sqrt{2}$ obey the triangle inequality. Using symmetry in B, C, we take cases:

- if $AB = 2020$, $AC = 2021$, then $BC = \sqrt{2}$.
- if $AB = 2020$, $BC = 2021$, then $AC = 2020 \pm 1010.5\sqrt{2}$.
- if $AB = 2021$, $BC = 2020$, then $AC = 2021 \pm 1010\sqrt{2}$.

Grouping conjugate pairs, the answer is $\sqrt{2} + 2 \cdot 2020 + 2 \cdot 2021 = 8082 + \sqrt{2}$.

§25 Mock AIME 2018/3

Twelve points are chosen uniformly and at random on the circumference of a circle. The probability there exists a diameter $\overline{MN}$ such that all twelve points lie on the same side of $\overline{MN}$ is $\frac{p}{q}$ for relatively prime integers p and q. Find $p + q$.

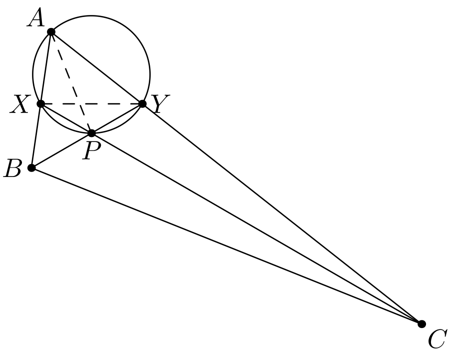
Answer. 515

Label the points $A_1, A_2, \dots, A_{12}$. Assume A_1 is the leftmost point. Because $\overline{MN}$ is the diameter of the circle, we want on the other points to be in the same semicircle. This occurs with probability $\frac{1}{2^{11}}$, because there are 11 other points. But, to select A_1, we have 12 options, so the probability in question is $\frac{12}{2^{11}} = \frac{3 \cdot 2^2}{2^{11}} = \frac{3}{2^9} = \frac{3}{512}$, and the answer is $3 + 512 = 515$.

§26 Mock AIME 2018/6

In triangle ABC, X lies on $\overline{AB}$ and Y lies on $\overline{AC}$ such that $\overline{BY}$ bisects $\angle ABC$ and $\overline{CX}$ bisects $\angle ACB$. Segments BY and CX intersect at a point P. Suppose that P lies on the circumcircle of triangle $\triangle AXY$. If $AX = 15$ and $AY = 24$, find AP^2.

Answer. 507



Notice that

$$180 = \angle A + \angle XPY = \angle A + \angle BPC = \angle A + \left(90 + \frac{\angle A}{2}\right) = 90 + \frac{3\angle A}{2},$$

whence $\angle A = 60$.

Since P is the incenter of $\triangle ABC$, $\overline{AP}$ is the angle bisector of $\angle A$. Then,

$$\angle PAX = \angle PAY = \angle PXY = \angle PYX = 30^\circ.$$

By the Law of Cosines on $\triangle AXY$, $XY = 21$. Since $\triangle PXY$ is an isosceles triangle with an angle of 120°, $PX = PY = 7\sqrt{3}$. Then, by Ptolemy's Theorem on quadrilateral $AXPY$,

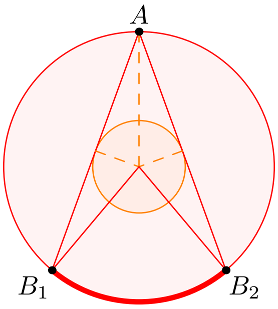
$$24 \cdot 7\sqrt{3} + 15 \cdot 7\sqrt{3} = 21 \cdot AP \implies AP = 13\sqrt{3},$$

so $AP^2 = 507$.

§27 Mock AIME 2019/5

Points A and B are randomly and uniformly chosen on the circumference of the circle $x^2 + y^2 = 1$. Find the expected number of ordered pairs of real numbers (p, q) such that the point (p, q) lies on line AB and there exists an integer $1 \le k \le 45$ such that $p^2 + q^2 = \sin^2(k^\circ)$.

Answer. 023



Let O denote the origin. Suppose that ω_k denotes the circle centered at O with radius $\sin k$. The problem is simply asking for the total number of times AB intersects ω_k for all integers $1 \le k \le 45$. We look at each k independently. On the unit circle, fix point A first.

Consider the locus of all B such that $\overline{AB}$ intersects ω_k. It is easy to check that this is the arc $\widehat{B_1B_2}$ not containing A such that $\overline{AB_1}$ and $\overline{AB_2}$ are tangent to ω_k. This is because for every point P on ω_k, the ray AP lies within $\angle B_1AB_2$, and every ray within $\angle B_1AB_2$ must intersect ω_k.

If X denotes the point where $\overline{AB_1}$ touches ω_k, it is easy to see that

$$\widehat{B_1B_2} = 2\angle B_1AB_2 = 4\angle XAO = 4 \arcsin\left(\frac{OX}{OA}\right) = 4k.$$

Hence, the probability a given line AB intersects ω_k is $\frac{4k}{360} = \frac{k}{90}$. The probability that $\overline{AB}$ is tangent to ω_k is infinitesimal and negligible, and if $\overline{AB}$ intersects ω_k but is not tangent to ω_k, then AB intersects ω_k at two points. Hence, the expected number of times $\overline{AB}$ intersects ω_k is $\frac{2k}{90} = \frac{k}{45}$. Summing over all such k, the answer is

$$\sum_{i=1}^{45} \frac{k}{45} = \frac{45 \cdot 46}{2 \cdot 45} = 23,$$

and we are done.

§28 CIME II 2020/6

An infinite number of buckets, labeled 1, 2, 3, …, lie in a line. A red ball, a green ball, and a blue ball are each tossed into a bucket, such that for each ball, the probability the ball lands in bucket k is 2^{-k}. Given that all three balls land in the same bucket B and that B is even, then the expected value of B can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer. 191

It is obvious that the probability all three balls land in bucket $2k$ is 64^{-k}. Then, the probability of the given condition occurring is

$$\sum_{k=1}^{\infty} \frac{1}{64^k} = \frac{64^{-1}}{1-64^{-1}} = \frac{1}{63}.$$

Hence, the expected value of B is

$$\left(\sum_{k=1}^{\infty} \frac{2k}{64^k}\right) \Big/ \frac{1}{63} = 126\left(\sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \frac{k}{64^k}\right) = 126\left(\sum_{j=1}^{\infty} \frac{64^{-j}}{1-64^{-1}}\right) = 126\left(\frac{63^{-1}}{63/64}\right) = \frac{128}{63},$$

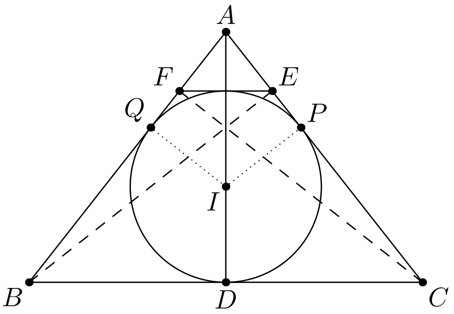
and the requested sum is $128 + 63 = 191$.

§29 CMC ARML 2019 I6

Let $\triangle ABC$ be an isosceles triangle with $AB = AC = 1$, and let E and F be the feet of the altitudes from B and C to sides AC and AB, respectively. If line EF is tangent to the incircle of $\triangle ABC$, compute the perimeter of $\triangle ABC$.

Answer. $1+\sqrt{5}$

Let $x = \sin\frac{A}{2}$, and denote by h_A and r the length of the A-altitude and the inradius, respectively. Furthermore let I denote the incenter and $\triangle DPQ$ the contact triangle of $\triangle ABC$.



Notice that since $\triangle AFE \sim \triangle ABC$ with scale factor $\cos A : 1$, we have that $1-\cos A = \frac{2r}{h_A}$. Clearly $BD = DC = x$, so the semiperimeter of $\triangle ABC$ is $1+x$, and $AP = AQ = 1-x$. Then, $r = AP\tan\frac{A}{2} = (1-x)\tan\frac{A}{2}$. Since $AD = \cos\frac{A}{2}$,

$$2x^2 = 1-\cos A = \frac{2(1-x)\tan\frac{A}{2}}{\cos\frac{A}{2}} = \frac{2x(1-x)}{\cos^2\frac{A}{2}} = \frac{2x(1-x)}{1-x^2}.$$

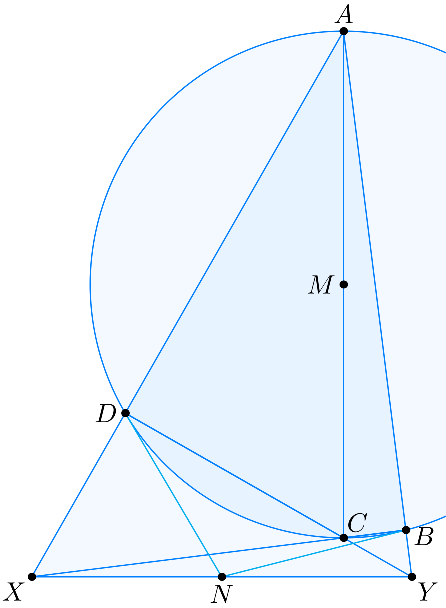
Since $x \neq 0$, we can simplify this to $x = \frac{1}{1+x}$, so $x^2+x-1=0$. As $-1 \le x \le 1$, $x = \frac{\sqrt{5}-1}{2}$, and the perimeter of $\triangle ABC$ is $2x+2 = 1+\sqrt{5}$, the answer.

§30 Mock AIME 2019/6

In cyclic quadrilateral $ABCD$, $AB = 8$, $BC = 1$, $CD = 4$, and $DA = 7$. If M denotes the midpoint of $\overline{AC}$, X the intersection of lines AD and BC, Y the intersection of lines AB and CD, and N the midpoint of $\overline{XY}$, then MN can be expressed in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime. Find $p+q+r$.

Answer. 078

First solution It is easy to determine that $\overline{AC}$ is a diameter of $(ABCD)$, and that $AC = \sqrt{65}$. The key observation is that A, X, Y, C form an orthocentric system, a corollary of which is that $\overline{AC} \perp \overline{XY}$.



First, let $XC = s, XD = t, YC = u, YB = v$. Then, since $\triangle XAB \sim \triangle XCD$, we have that

$$2 = \frac{s+1}{t} = \frac{t+7}{s}.$$

Solving, $s = 5$ and $t = 3$. Similarly, $\triangle YAD \sim \triangle YCB$, so

$$7 = \frac{u+4}{v} = \frac{v+8}{u}.$$

Solving, $u = \frac{5}{4}$ and $v = \frac{3}{4}$. It follows that by the Pythagorean theorem on either $\triangle XDY$ or $\triangle XBY$, we compute that $XY = \frac{3\sqrt{65}}{4}$.

It is well-known that $\overline{NB}$ and $\overline{ND}$ are tangent to $(ABCD)$. Then,

$$\begin{aligned} MN^2 - \left(\frac{1}{2}AC\right)^2 &= \operatorname{Pow}_{(ABCD)}(N) = NB^2 = \left(\frac{1}{2}XY\right)^2 \\ \implies MN^2 &= \left(\frac{\sqrt{65}}{2}\right)^2 + \left(\frac{3\sqrt{65}}{8}\right)^2 = \frac{25 \cdot 65}{64} \\ \implies MN &= \frac{5\sqrt{65}}{8}, \end{aligned}$$

and the requested sum is $5 + 65 + 8 = 78$.

Second solution Like above, note that $\overline{AC}$ is a diameter of $(ABCD)$, and that $AC = \sqrt{65}$. Furthermore, C is the orthocenter of $\triangle AXY$. Let $Z = \overline{AC} \cap \overline{XY}$ be the foot from A to $\overline{XY}$, and let $\alpha = \angle CAD$ and $\beta = \angle CAB$. It is easy to check that $\tan\alpha = \frac{4}{7}$ and $\tan\beta = \frac{1}{8}$. Then,

$$\tan(\alpha+\beta) = \frac{\frac{4}{7}+\frac{1}{8}}{1-\frac{4}{7}\cdot\frac{1}{8}} = \frac{3}{4}.$$

Hence,

$$AX = AB\sec(\alpha+\beta) = 8\cdot\frac{5}{4} = 10$$

and similarly $AY = \frac{35}{4}$. Now,

$$AZ = AX\cos\alpha = 10\cdot\frac{7}{\sqrt{65}} = \frac{14\sqrt{65}}{13}$$

and

$$XZ = AX\sin\alpha = 10\cdot\frac{4}{\sqrt{65}} = \frac{8\sqrt{65}}{13}.$$

Similarly, $YZ = \frac{7\sqrt{65}}{52}$. It follows that

$$MZ = AZ - AM = \frac{14\sqrt{65}}{13} - \frac{\sqrt{65}}{2} = \frac{15\sqrt{65}}{26}.$$

Furthermore,

$$NZ = XZ - \frac{XY}{2} = \frac{8\sqrt{65}}{13} - \frac{3\sqrt{65}}{8} = \frac{25\sqrt{65}}{104}.$$

It follows that

$$MN = \sqrt{MZ^2+NZ^2} = \frac{5\sqrt{65}}{104}\sqrt{12^2+5^2} = \frac{5\sqrt{65}}{8},$$

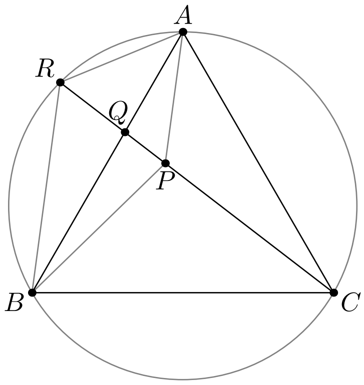
and the requested sum is $5+65+8 = 78$.

§31 NEMO 2020 I20

Let ABC be an equilateral triangle and P a point in its interior obeying $AP = 5$, $BP = 7$, $CP = 8$. Line CP intersects $\overline{AB}$ at Q. Compute AQ.

Answer. $\frac{5\sqrt{129}}{13}$

Assume $\angle CAB$ is oriented counterclockwise. Let R be the image of P under 60° clockwise rotation at A. Since $AP = AR = 5$ and $\angle PAR = 60^\circ$, $\triangle APR$ is equilateral; i.e. $PR = 5$. Since $BR = 8$, $BP = 7$, we have $\angle BRP = 60^\circ$, say by cosine law on $\triangle BPR$.



Now $\angle APR = 60^\circ$ and $\angle APC = 120^\circ$, so C, P, R collinear; thus $\overline{RC}$ bisects $\angle ARB$ and

$$\frac{AQ}{BQ} = \frac{AR}{BR} = \frac{5}{8}.$$

Finally, cosine law on $\triangle ARB$ gives $AB = \sqrt{129}$, so $AQ = \frac{5\sqrt{129}}{13}$ as desired.

§32 CIME II 2020/8

A committee has an oligarchy, consisting of $A\%$ of the members of the committee. Suppose that $B\%$ of the work is done by the oligarchy. If the average amount of work done by a member of the oligarchy is 16 times the amount of work done by a nonmember of the oligarchy, find the maximum possible value of $B - A$.

Answer. 060

Scale A and B down by a factor of 100. Assume that the committee has 1 member who can be split into many pieces, and the committee does 1 joule of work. Then

$$16 = \frac{B \text{ joules}}{A \text{ people}} \div \frac{(1-B) \text{ joules}}{(1-A) \text{ people}} = \frac{B}{1-B} \cdot \frac{1-A}{A}.$$

Rearranging,

$$B - AB = 16A - 16AB \implies A = \frac{B}{16-15B},$$

so we want to maximize $f(x) = x - \frac{x}{16-15x}$ over $(0,1)$. Assume the expression equals T. Rearranging,

$$0 = 15x^2 - 15(T+1)x + 16T.$$

To maximize T, we set the discriminant equal to 0: this gives

$$15^2(T+1)^2 = 15 \cdot 64T \implies 0 = T^2 - \frac{34}{15}T + 1.$$

By Po's quadratic method, $T = \frac{17}{15} - \frac{8}{15} = \frac{3}{5}$. The answer is 060.

§33 CIME II 2020/9

Let $f(x) = x^2 - 2$. There are N real numbers x such that

$$\underbrace{f(f(\dots f}_{2019 \text{ times}}(x)\dots)) = \underbrace{f(f(\dots f}_{2020 \text{ times}}(x)\dots)).$$

Find the remainder when N is divided by 1000.

Answer. 433

The fixed points of f are 2 and -1, so we want $f^{2019}(x) \in \{2,-1\}$. Let a_n be the number of solutions to $f^n(x) \in \{2,-1\}$, and let $b_n = a_n - a_{n-1}$.

By arrows, all x with $f^n(x) \in \{2,-1\}$ for some x must satisfy $x \geq -2$. Otherwise $f(x) > 2$ and f keeps on increasing. Note that for each x_0 such that $f^n(x_0) \in \{2,-1\}$ but $f^{n-1}(x_0) \notin \{2,-1\}$, the solutions to $f(x) = x_0$ satisfy $f^{n+1}(x_0) \in \{2,-1\}$ but $f^n(x_0) \notin \{2,-1\}$. For each $x_0 \notin \{2,-1\}$, $f(x) = x_0$ has two distinct roots.

Hence $b_{n+1} = 2b_n$ for $n \geq 2$. In otherwords, $a_{n+1} = 3a_n - 2a_{n-1}$. By the base case $a_1 = 4$ and $a_2 = 7$, standard methods yield $a_n = 3 \cdot 2^{n-1} + 1$. Hence $N = 3 \cdot 2^{2018} + 1$, and CRT/Euler give an answer of 433.

§34 Mock AIME 2019 Tiebreaker/V

Call a positive integer *palatable* if when expressed in binary, each contiguous block of zeros that is not a subsequence of another contiguous block of zeros has even length, and each contiguous block of ones that is not a subsequence of another contiguous block of ones has odd length. For example, $57 = 111001_2$ is palatable while $69 = 1000101_2$ is not. Find the number of palatable positive integers N such that $2^{18} < N < 2^{19}$.

Answer. 692

Let a_k be the number of palatable integers with k digits in base 2 that end in 0, and b_k the number that end in 1. If the number T ends in 0 in binary, we can append two 0's or one 1 to form another palatable integer, while if T ends in 1 in binary, we can append two 0's or two 1's to form another. Hence, $a_k = a_{k-2} + b_{k-2}$ and $b_k = a_{k-1} + b_{k-2}$. It is easy to check that $a_1 = 0$, $a_2 = 0$, $b_1 = 1$, and $b_2 = 0$. Then,

k	a_k	b_k
1	0	1
2	0	0
3	1	1
4	0	1
5	2	1
6	1	3
7	3	2
8	4	6
9	5	6
10	10	11
11	11	16
12	21	22
13	27	37
14	43	49
15	64	80
16	92	113
17	144	172
18	205	257
19	316	377

Since 2^{18} is palatable, the answer is $a_{19} + b_{19} - 1 = 692$.

§35 CIME II 2019/5

Let $a = 5 + 2i$ and $b = 18 + 13i$, where $i = \sqrt{-1}$. Suppose that z and ω are complex numbers such that

$$\left(z + \frac{1}{z}\right) + \left(\omega + \frac{1}{\omega}\right) = \left(a + \frac{1}{a}\right) \times \left(b + \frac{1}{b}\right), \text{ and}$$
$$\left(z + \frac{1}{z}\right) \times \left(\omega + \frac{1}{\omega}\right) = \left(a^2 + \frac{1}{a^2}\right) + \left(b^2 + \frac{1}{b^2}\right).$$

Then, the largest possible value of $|z + \omega|$ can be expressed as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Answer. 047

Notice that

$$\begin{aligned}\left(z+\frac{1}{z}\right)+\left(\omega+\frac{1}{\omega}\right)&=\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\\&=\frac{a}{b}+\frac{b}{a}+ab+\frac{1}{ab}\\&=\left(ab+\frac{1}{ab}\right)+\left(\frac{a}{b}+\frac{b}{a}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\left(z+\frac{1}{z}\right)\left(\omega+\frac{1}{\omega}\right)&=\left(a^2+\frac{1}{a^2}\right)+\left(b^2+\frac{1}{b^2}\right)\\&=\frac{a^2+b^2}{a^2b^2}+a^2+b^2\\&=\left(1+\frac{1}{a^2b^2}\right)(a^2+b^2)\\&=\left(ab+\frac{1}{ab}\right)\left(\frac{a}{b}+\frac{b}{a}\right).\end{aligned}$$

By Vieta's, WLOG let $z+\frac{1}{z}=ab+\frac{1}{ab}$ and $\omega+\frac{1}{\omega}=\frac{a}{b}+\frac{b}{a}$. Then, $z\in\{ab,\frac{1}{ab}\}$ and $\omega\in\{\frac{a}{b},\frac{b}{a}\}$. Notice that

$$ab=(5+2i)(18+13i)=(90-26)+(65+36)i=64+101i$$

and

$$\frac{b}{a}=\frac{18+13i}{5+2i}=\frac{(18+13i)(5-2i)}{29}=\frac{116+29i}{29}=4+i.$$

It is easy to see that the maximum possible value of $|z+\omega|$ occurs when $z=ab$ and $\omega=\frac{b}{a}$. Hence,

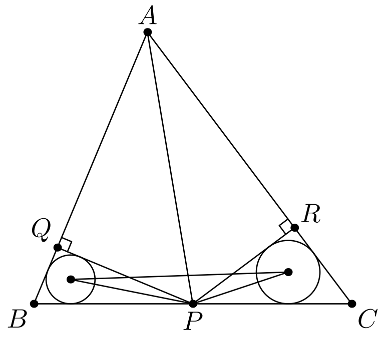
$$\max(|z+\omega|)=|68+102i|=34\,|2+3i|=34\sqrt{13},$$

and the requested sum is $34+13=47$.

§36 Mock AIME 2018/12

In triangle ABC, $AB=13$, $BC=14$, $CA=15$, and a point P lies on $\overline{BC}$. Let Q be the foot of the perpendicular from P to $\overline{AB}$ and R be the foot of the perpendicular from P to $\overline{AC}$. Suppose I_B and I_C are the incenters of triangles $\triangle PBQ$ and $\triangle PCR$, respectively. Then the maximum possible area of $\triangle PI_BI_C$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer. 457



First, we will find a closed form for the area of $\triangle PI_BI_C$. Notice that $\overline{PI_B}$ bisects $\angle BPQ$ and $\overline{PI_C}$ bisects $\angle CPR$, so

$$\angle I_BPI_C = 180 - \frac{(90 - \angle B) + (90 - \angle C)}{2} = 180 - \frac{\angle A}{2}$$

Since $\angle I_BPI_C$ is fixed, it suffices to maximize $PI_B \cdot PI_C$.

Consider $\triangle PBI_B$. Notice that $\angle BI_BP = 90 + \frac{\angle BQP}{2} = 135^\circ$. By the Law of Sines,

$$\frac{PI_B}{\sin \angle PBI_B} = \frac{BP}{\sin \angle BI_BP} \implies PI_B = \sqrt{2} \cdot BP \cdot \sin\left(\frac{\angle B}{2}\right).$$

Similarly, we know $PI_C = \sqrt{2} \cdot CP \cdot \sin\left(\frac{\angle C}{2}\right)$. It follows that

$$\begin{aligned}
[PI_BI_C] &= \frac{1}{2} \cdot PI_B \cdot PI_C \cdot \sin \angle I_BPI_C \\
&= \frac{1}{2} \cdot \sqrt{2} \cdot BP \cdot \sin\left(\frac{\angle B}{2}\right) \cdot \sqrt{2} \cdot CP \cdot \sin\left(\frac{\angle C}{2}\right) \cdot \sin\left(\frac{\angle A}{2}\right) \\
&= BP \cdot CP \cdot \sin\left(\frac{\angle A}{2}\right) \sin\left(\frac{\angle B}{2}\right) \sin\left(\frac{\angle C}{2}\right).
\end{aligned}$$

Since the angles are fixed, it suffices to maximize $BP \cdot CP$. Since $BP + CP = 14$, by AM-GM the maximum possible value of $BP \cdot CP$ occurs when $BP = CP = 7$.

It follows that

$$[PI_BI_C] = 49 \sin\left(\frac{\angle A}{2}\right) \sin\left(\frac{\angle B}{2}\right) \sin\left(\frac{\angle C}{2}\right)$$

Suppose X is the foot of the perpendicular from A to $\overline{BC}$. It is well known that $AX = 12$, $BX = 5$, and $CX = 9$. It is easy to see by the Cosine Addition Formula that $\cos \angle A = \frac{33}{65}$. Then, we can apply the Half Angle Formulas to see that

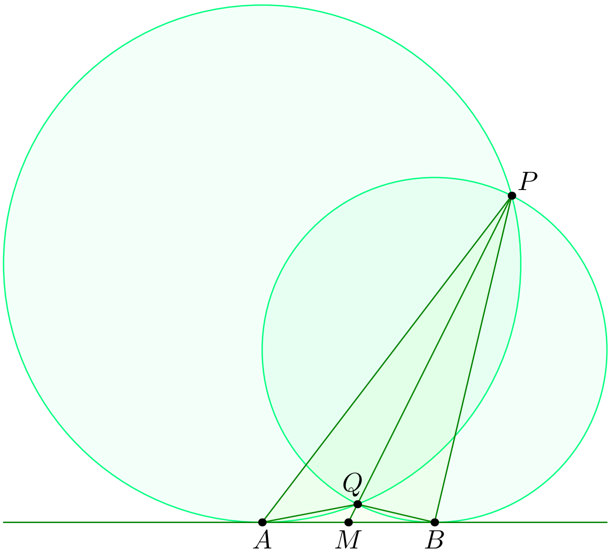
$$\begin{aligned}
[PI_BI_C] &= 49\sqrt{\frac{\left(1 - \frac{33}{65}\right)\left(1 - \frac{5}{13}\right)\left(1 - \frac{3}{5}\right)}{2 \cdot 2 \cdot 2}} \\
&= 49\sqrt{\frac{\frac{32}{65} \cdot \frac{8}{13} \cdot \frac{2}{5}}{2 \cdot 2 \cdot 2}} \\
&= 49\sqrt{\frac{8^2}{65^2}} = \frac{49 \cdot 8}{65} = \frac{392}{65}.
\end{aligned}$$

The answer is then $392 + 65 = 457$.

§37 Mock AIME 2019/10

Suppose that circles Ω_1 and Ω_2 intersect at P and Q, and that line AB is tangent to Ω_1 and Ω_2 at A and B, respectively, such that Q is closer to $\overline{AB}$ than P. If $AB = 2$, $PA = 20$, and $PB = 19$, then $QA \cdot QB$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

Answer. 519



First, remark that if in any triangle XYZ, $YZ = 2$ and N denotes the midpoint of $\overline{YZ}$, then by Stewart's Theorem,

$$XN^2 = \frac{XY^2 + XZ^2}{2} - 1.$$

Let $M = \overline{AB} \cap \overline{PQ}$. Then, $MA^2 = MP \cdot MQ = MB^2$, so M is the midpoint of $\overline{AB}$. By the Tangency Criterion, $\triangle MAQ \sim \triangle MPA$ and $\triangle MBQ \sim \triangle MPB$. Hence,

$$\frac{PA}{QA} = \frac{MP}{MA} = \frac{MP}{MB} = \frac{PB}{QB}.$$

It follows that there exists a real number t such that $QA = 20t$ and $QB = 19t$. By Power of a Point from M, $MP \cdot MQ = 1$. If $u = \frac{20^2+19^2}{2} = \frac{761}{2}$, using our remark,

$$1 = MP^2 \cdot MQ^2 = \left(\frac{20^2+19^2}{2} - 1\right)\left(t^2 \cdot \frac{20^2+19^2}{2} - 1\right) = (u-1)(t^2u-1),$$

whence

$$1 + \frac{1}{t^2} = u = \frac{761}{2} \implies t^2 = \frac{2}{759} \implies QA \cdot QB = 380t^2 = \frac{760}{759},$$

and the requested remainder is $760 + 759 \equiv 519 \pmod{1000}$.

§38 Mock AIME 2019/11

Adam, Bob, and Charlie each flip a coin every day, starting from Day 1, until all three of them have flipped heads at least once. The last of them to flip heads for the first time does so on Day X. The probability that X is even can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer. 068

If k remains fixed, note that

$$\begin{aligned}\mathbb{P}(X > k) &= 1 - \mathbb{P}(\text{All three have flipped heads in the first } k \text{ days}) \\ &= 1 - \mathbb{P}(\text{Adam has flipped heads in the first } k \text{ days})^3 \\ &= 1 - \big(1 - \mathbb{P}(\text{Adam has not flipped heads in the first } k \text{ days})\big)^3 \\ &= 1 - \left(1 - \frac{1}{2^k}\right)^3.\end{aligned}$$

It follows that

$$\mathbb{P}(X=k)=\mathbb{P}(X>k-1)-\mathbb{P}(X>k)=\left(1-\frac{1}{2^k}\right)^3-\left(1-\frac{1}{2^{k-1}}\right)^3.$$

Hence,

$$\begin{aligned}\mathbb{P}(2\mid X)=\sum_{i=1}^{\infty}\mathbb{P}(X=2i)&=\sum_{i=1}^{\infty}\left(\left(1-\frac{1}{2^{2i}}\right)^3-\left(1-\frac{1}{2^{2i-1}}\right)^3\right)\\ &=\sum_{i=1}^{\infty}\Bigg(-3\left(-\frac{1}{2^{2i-1}}+\frac{1}{2^{2i}}\right)+3\left(-\left(\frac{1}{2^{2i-1}}\right)^2+\left(\frac{1}{2^{2i}}\right)^2\right)\\ &\qquad-\left(-\left(\frac{1}{2^{2i-1}}\right)^3+\left(\frac{1}{2^{2i}}\right)^3\right)\Bigg)\\ &=-3\left(\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^k\right)+3\left(\sum_{k=1}^{\infty}\left(-\frac{1}{2^2}\right)^k\right)-\sum_{k=1}^{\infty}\left(-\frac{1}{2^3}\right)^k\\ &=-3\left(\frac{-1/2}{3/2}\right)+3\left(\frac{-1/4}{5/4}\right)-\frac{-1/8}{9/8}=1-\frac{3}{5}+\frac{1}{9}=\frac{23}{45},\end{aligned}$$

and the requested sum is $23+45=68$.

§39 CIME II 2019/12

Max is playing a video game with 99 levels, labeled $1,2,\ldots,99$. Whenever Max completes a level, he begins the next one immediately. However, for all $1\le n\le 99$, Max fails the n^{th} level with probability $(n+1)^{-2}$. Whenever he fails a level, he quits for the day and attempts the level again the next day. If Max first attempts the first level on Day 1 and completes the 99^{th} level on Day K, then the expected value of K can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p+q$ is divided by 1000.

Answer. 349

Suppose that Max attempts only one free throw a day. Then, the expected number of days it takes Max to complete level n is

$$\left(1-\frac{1}{(n+1)^2}\right)^{-1}=\left(\frac{n(n+2)}{(n+1)^2}\right)^{-1}=1+\frac{1}{n(n+2)}=1+\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right).$$

Thus, the expected number of days it takes Max to complete all 99 levels is

$$\begin{aligned}\sum_{n=1}^{99}\left(1-\frac{1}{(n+1)^2}\right)^{-1}&=\sum_{n=1}^{99}\left(\frac{n(n+2)}{(n+1)^2}\right)^{-1}\\ &=99+\frac{1}{2}\sum_{n=1}^{99}\left(\frac{1}{n}-\frac{1}{n+2}\right)\\ &=99+\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{100}-\frac{1}{101}\right)\\ &=99+\frac{14949}{20200}.\end{aligned}$$

However, we overcount by 98 days, so the expected value of K is $1+\frac{14949}{20200}=\frac{35149}{20200}$, and the requested remainder is $35149+20200\equiv 349 \pmod{1000}$.

§40 CIME I 2020/11

An *excircle* of a triangle is a circle tangent to one of the sides of the triangle and the extensions of the other two sides. Let ABC be a triangle with $\angle ACB = 90^\circ$ and let r_A, r_B, r_C denote the radii of the excircles opposite to A, B, C, respectively. If $r_A = 9$ and $r_B = 11$, then r_C can be expressed in the form $m + \sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

Answer. 209

Let $a = BC$, $b = CA$, $c = AB$, $s = \frac{a+b+c}{2}$, and K be the area of $\triangle ABC$. Remark that since $\angle ACB = 90^\circ$, if the C-excircle touches $\overline{AB}, \overline{BC}, \overline{CA}$ at C', A', B', respectively, then $CA'I_CB'$ is a square, so $r_C = I_AA' = CA' = s$. It is known that

$$K = r_A(s-a) = r_B(s-b) = r_C(s-c).$$

Notice that

$$\begin{aligned} s(s-c) &= \frac{(a+b+c)(a+b-c)}{4} = \frac{(a+b)^2 - c^2}{4} \\ &= \frac{(a+b)^2 - a^2 - b^2}{4} = \frac{ab}{2} = K, \end{aligned}$$

and by Heron's $(s-a)(s-b) = K$ as well. Check that

$$r_A + r_B = \frac{K}{s-a} + \frac{K}{s-b} = K\left(\frac{c}{(s-a)(s-b)}\right) = c.$$

Furthermore,

$$ab = 2K = 2 \cdot \frac{K}{s-a} \cdot \frac{K}{s-b} = 2r_Ar_B.$$

Hence, $a^2 + b^2 = (r_A + r_B)^2$ and $2ab = 4r_Ar_B$. Adding, $a + b = \sqrt{(r_A+r_B)^2 + 4r_Ar_B}$, and it readily follows that

$$r_C = \frac{a+b+c}{2} = \frac{r_A + r_B + \sqrt{(r_A+r_B)^2 + 4r_Ar_B}}{2},$$

which evaluates to $10 + \sqrt{199}$. The requested sum is $10 + 199 = 209$.

§41 CIME II 2019/11

In triangle ABC with incenter I, $AB = 4$, $BC = 5$, and $CA = 6$. If lines AI and BI meet the circumcircle of $\triangle ABC$ again at S and L, respectively, and $\overline{LB}$ and $\overline{LS}$ intersect $\overline{AC}$ at D and E, respectively, then the square of the area of quadrilateral $SIDE$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer. 121

To begin, let L' be the reflection of A over the perpendicular bisector of $\overline{BC}$, so that $\triangle ABC \cong \triangle L'CB$. However, by Ptolemy's on $ABCL'$,

$$5AL' + 4 \cdot 4 = 6 \cdot 6 \implies AL' = 4.$$

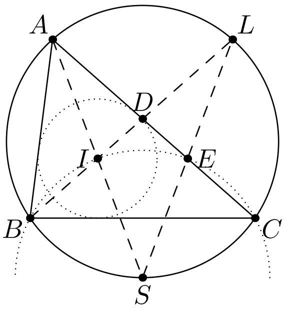
However, it follows that $AB = AL' = CL' = 4$, so $L = L'$; that is, $ABCL$ is an isosceles trapezoid. Now, by the Incenter-Excenter Lemma, $SB = SI = SC = d$ for some d. By Ptolemy's on $ABSC$,

$$4 \cdot d + 6 \cdot d = 10 \cdot AS \implies d = \frac{AS}{2},$$

whence I is the midpoint of $\overline{AS}$.

The semiperimeter s of $\triangle ABC$ is clearly $15/2$. It follows by Heron's and $K = rs$, where r denotes the inradius of $\triangle ABC$, that

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \div \frac{15}{2}} = \frac{\sqrt{7}}{2},$$



If F denotes the tangency point between the incircle and $\overline{AB}$, we can compute that $AF = s - a = 5/2$. Then, by the Pythagorean Theorem on $\triangle AFI$, $AI^2 = \sqrt{(s-a)^2 + r^2} = 2\sqrt{2}$.

Now, since I and E are the midpoints of $\overline{SA}$ and $\overline{SL}$, respectively, D is the centroid of $\triangle ASL$, so $[SIDE] = [ASL]/3$. However, since $AS = 2AI = 4\sqrt{2}$ and $AL = 4$, the distance from S to AL is $\sqrt{(4\sqrt{2})^2 - 2^2} = 2\sqrt{7}$, whence

$$[SIDE] = \frac{[ASL]}{3} = \frac{\frac{1}{2} \cdot 4 \cdot 2\sqrt{7}}{3} = \frac{4\sqrt{7}}{3} \implies [SIDE]^2 = \frac{112}{9},$$

and the requested sum is $112 + 9 = 121$.

> **Remark** (Alternative solution)**.** By the Angle Bisector Theorem, $AD = 8/3$ and $DC = 10/3$. However,
>
> $$BD^2 = BA \cdot BC - DA \cdot DC = 20\left(1 - \left(\frac{2}{3}\right)^2\right) = \frac{100}{9},$$
>
> so $\triangle BDC$ is isosceles. Then, $AL = LC = AB = 4$, and we may proceed as above.

§42 Mock AIME 2018/15

Let a, b, c, and d be positive real numbers such that

$$195 = a^2 + b^2 = c^2 + d^2 = \frac{13\left(ac + bd\right)^2}{13b^2 - 10bc + 13c^2} = \frac{5\left(ad + bc\right)^2}{5a^2 - 8ac + 5c^2}.$$

Find the greatest integer that does not exceed $a + b + c + d$.

Answer. 034

Clearly, an Algebra Bash would be catastrophic. Instead, we present a geometric approach.

Lemma

In cyclic quadrilateral $ABCD$ with circumradius R, the area K of $ABCD$ is

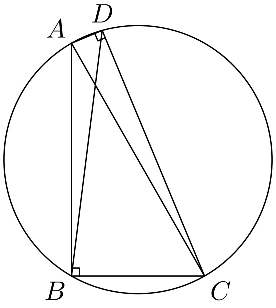
$$K = 2R^2 \sin A \sin B \sin \theta,$$

where θ is either one of the angles formed by the diagonals AC and BD of quadrilateral $ABCD$.

Proof. Suppose $\widehat{AB} = w$, $\widehat{BC} = x$, $\widehat{CD} = y$, and $\widehat{DA} = z$. By the Extended Law of Sines, $BD = 2R \sin A$ and $AC = 2R \sin B$. It follows that

$$K = \frac{1}{2} \cdot BD \cdot AC \cdot \sin\theta = 2R^2 \sin A \sin B \sin\theta,$$

and our lemma has been proven. □



Consider a cyclic quadrilateral $ABCD$ with $\angle ABC = \angle CDA = 90^\circ$ and $AB = a$, $BC = b$, $CD = c$, and $DA = d$. Then, the diameter of the circumcircle of $ABCD$ is $AC = \sqrt{195}$.

By Ptolemy's Theorem, $ac + bd = \sqrt{195} \cdot BD$. Substituting into the given equation gives

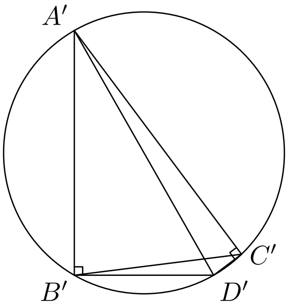
$$195 = \frac{13(ac+bd)^2}{13b^2 - 10bc + 13c^2} = \frac{13 \cdot 195 \cdot BD^2}{13b^2 - 10bc + 13c^2} = \frac{195 \cdot BD^2}{b^2 - \frac{10}{13}bc + c^2}.$$

We then have $BD^2 = b^2 + c^2 - 2bc \cdot \frac{5}{13}$. This is the Law of Cosines on $\triangle BCD$. It is easy to see that $\cos\angle BCD = \frac{5}{13}$. It follows that $\sin\angle BCD = \frac{12}{13}$. Then, by our Lemma, the area of $ABCD$ is

$$K = 2 \cdot \left(\frac{\sqrt{195}}{2}\right)^2 \cdot 1 \cdot \frac{12}{13} \cdot \sin\theta = 90 \sin\theta.$$

By the Law of Sines on $\triangle BCD$, $BD = \frac{12\sqrt{195}}{13}$. By Ptolemy's, $ac + bd = 180$.

Suppose we define the lengths of the arcs as in the proof of our lemma. Then, $\theta = \frac{w+y}{2}$.



Now consider cyclic quadrilateral $A'B'D'C'$, with $\angle A'B'D' = \angle D'C'A' = 90^\circ$ and $A'B' = a$, $B'D' = b$, $D'C' = d$, and $C'A' = c$. It follows that $\widehat{A'B'} = w$, $\widehat{B'D'} = x$, $\widehat{D'C'} = z$, and $\widehat{C'A'} = y$. Also note that $A'D' = \sqrt{195}$. Notice that

$$\sin\theta = \sin\left(\frac{w+y}{2}\right) = \sin\left(\frac{x+z}{2}\right) = \sin\angle B'A'C'.$$

By Ptolemy's, $ad + bc = \sqrt{195}\cdot B'C'$. Substituting into the given equation gives

$$195 = \frac{5(ad+bc)^2}{5a^2-8ac+5c^2} = \frac{5\cdot 195\cdot B'C'^2}{5a^2-8ac+5c^2} = \frac{195\cdot B'C'^2}{a^2-\frac{8}{5}ac+c^2}.$$

We then have $B'C'^2 = a^2+c^2-2ac\cdot\frac{4}{5}$. Then, $\cos\angle B'A'C' = \frac{4}{5}$, and $\sin\theta = \sin\angle B'A'C' = \frac{3}{5}$. It follows that

$$K = 90\sin\theta = 54 \implies ab + cd = 108.$$

Note that $B'C' = \frac{3\sqrt{195}}{5}$ by the Law of Sines on $\triangle A'B'C'$. By Ptolemy's, $ad+bc = 117$. It follows that

$$\begin{aligned} a+b+c+d &= \sqrt{(a^2+b^2)+(c^2+d^2)+2(ac+bd)+2(ab+cd)+2(ad+bc)} \\ &= \sqrt{195+195+2\cdot 180+2\cdot 108+2\cdot 117} = \sqrt{1200} = 20\sqrt{3}, \end{aligned}$$

and the answer is $\lfloor 20\sqrt{3}\rfloor = 34$.

§43 CIME II 2021/14

Among all pairs of positive integers (x, y) such that $x+y$ divides 2020, randomly and uniformly select one of them. The expected value of $\gcd(x, y)$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m+n$ is divided by 1000.

Answer. 019

Let $\sigma_0(n)$ denote the number of divisors of n and $\sigma_1(n)$ the sum of the divisors of n. In what follows, $N = 2020$, $\sigma_0(N) = 12$, $\sigma_1(N) = 4284$. The number of such (x, y) is

$$\sum_{d\mid N}(d-1) = \sigma_1(N) - \sigma_0(N) = 4272,$$

so we turn to computing the sum of all $\gcd(x, y)$.

Consider this sum:

$$\sum_{x+y=n}\gcd(x,y) = \sum_{\substack{d\mid n\\ d\neq n}} d\varphi\left(\frac{N}{d}\right) = (\mathrm{id}*\varphi)(n) - n.$$

The desired sum of $\gcd(x, y)$ is

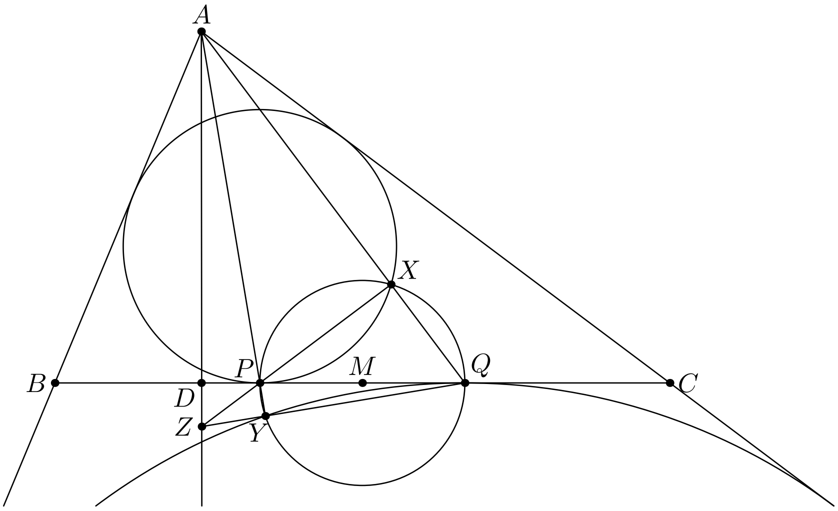
$$\begin{aligned} \sum_{n\mid N}\sum_{x+y=n}\gcd(x,y) &= \sum_{n\mid N}\left[(\mathrm{id}*\varphi)(n) - n\right] \\ &= (\mathrm{id}*\varphi*1)(N) - \sigma_1(N) \\ &= (\mathrm{id}*\mathrm{id})(N) - \sigma_1(N) \\ &= N\cdot\sigma_0(N) - \sigma_1(N) \\ &= 19956. \end{aligned}$$

The expected gcd is $\dfrac{19956}{4272} = \dfrac{1663}{356}$, and the requested sum is $1663 + 356 = 2019$.

§44 Mock AIME 2019 Tiebreaker/X

In triangle ABC, $AB = 26$, $BC = 42$, and $CA = 40$. Let ω be the incircle of $\triangle ABC$, and let ω_A be the circle tangent to segment BC and the extensions of lines AB and AC past B and C, respectively. Suppose that ω and ω_A are tangent to $\overline{BC}$ at P and Q, respectively, and that X and Y lie on ω and ω_A, respectively, such that $\angle AXP = \angle AYQ = 90^\circ$. If M is the midpoint of $\overline{BC}$ and Z is the intersection of $\overline{PX}$ and $\overline{QY}$, find MZ^2.

Answer. 130



It is well known that $MP = MQ$, so M is the center of Γ, the circle with diameter $\overline{PQ}$. It is well known that if P' and Q' denote the antipodes of P and Q, respectively, on ω and ω_A, respectively, then $P' \in \overline{AQ}$ and $Q' \in \overline{AP}$. Let Ω meet ω and ω_A again at $X' \neq P$ and $Y' \neq Q$, respectively. Since

$$\measuredangle PX'Q = 90^\circ = \measuredangle PX'P' = \measuredangle PX'A \text{ and } \measuredangle PY'Q = 90^\circ = \measuredangle Q'Y'Q = \measuredangle AY'Q,$$

we know that $X = X'$ and $Y = Y'$. Moreover, $X \in \overline{AQ}$ and $Y \in \overline{AZ}$.

Since $\measuredangle AXZ = \measuredangle AYZ = 90^\circ$, Q is the orthocenter of $\triangle APZ$, so $\overline{AZ} \perp \overline{BC}$. Then, let D be the foot from A to $\overline{BC}$, so that $D \in \overline{AZ}$.

Now, we compute some lengths. Scale down by a factor of 2, so that $AB = 13$, $BC = 21$, and $CA = 20$. It is not hard to check that $AD = 12$, $BD = 5$, and $CD = 16$. It follows that $[ABC] = 126$ and the semiperimeter is $s = 27$. Furthermore, $BP = s - AC = 7$, so $DP = 2$. Since $BM = \frac{21}{2}$, $MP = MQ = \frac{7}{2}$, $PQ = 7$, $DQ = 9$, and $AQ = 15$. By Power of a Point from Q on $(ADPX)$,

$$QD \cdot QP = QA \cdot QX \implies 9 \cdot 7 = 15 \cdot QX \implies QX = \frac{21}{5} \implies AX = \frac{54}{5}.$$

By Power of a Point from A on $(DXQZ)$,

$$AX \cdot AQ = AD \cdot AZ \implies \frac{54}{5} \cdot 15 = 12 \cdot AZ \implies AZ = \frac{27}{2} \implies DZ = \frac{3}{2}.$$

Finally, by the Pythagorean Theorem on $\triangle DZM$,

$$ZM = \sqrt{DM^2 + DZ^2} = \sqrt{\left(\frac{11}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \frac{\sqrt{130}}{2}.$$

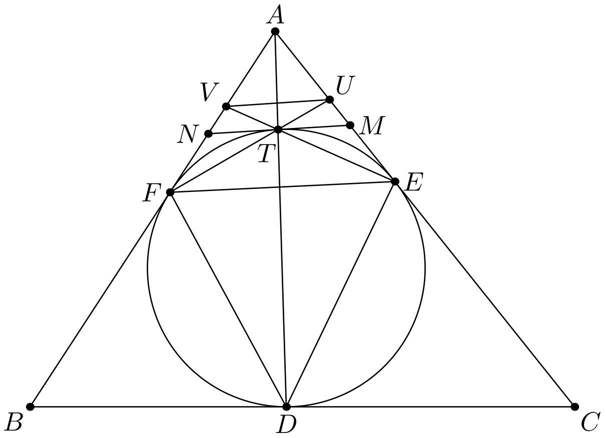
Restoring the factor of 2 yields $ZM^2 = 130$, the answer.

§45 Mock AIME 2019 Tiebreaker/Z

In triangle ABC, $AB = 14$, $BC = 17$, and $CA = 15$. The incircle of $\triangle ABC$ touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively; and $\overline{AD}$ meets the incircle of $\triangle ABC$ again at T. Suppose that points M and U lie on $\overline{AC}$ and points N and V lie on $\overline{AB}$ such that $\overline{MN}$ is tangent to the incircle of $\triangle ABC$ at T, and $\overline{EV}$ and $\overline{FU}$ intersect at T. Then, there exist relatively prime positive integers p and q such that $\frac{MU}{NV} = \frac{p}{q}$. Find $p + q$.

Answer. 449

Let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a + b + c)$.



Denote $P = \overline{BC} \cap \overline{EF}$. Since $DETF$ is a harmonic quadrilateral, $P \in \overline{MN}$. By Brianchon's Theorem on $BDCMTN$ and $BCEMNF$, there exists a common point S on $\overline{BM}$, $\overline{CN}$, $\overline{DT}$, and $\overline{EF}$. Now, note that

$$-1 = (A, S; T, D) \overset{P}{=} (A, F; N, B) \tag{1}$$

and

$$-1 = (D, T; E, F) \overset{T}{=} (A, N; V, F). \tag{2}$$

By (1), we can deduce that

$$\frac{NA}{NF} = \frac{BA}{BF} = \frac{c}{s-b}.$$

It follows that

$$\frac{VA}{VN} = \frac{FA}{FN} = 1 + \frac{NA}{NF} = \frac{s-b+c}{s-b}.$$

Moreover we have that

$$\frac{NA}{AF} = \frac{c}{s-b+c} \implies NA = \frac{c(s-a)}{s-b+c}.$$

Thus,

$$\frac{NV}{NA} = \frac{s-b}{2(s-b)+c},$$

so

$$NV = \frac{c(s-a)(s-b)}{\big(2(s-b)+c\big)(s-b+c)} = \frac{14 \cdot 6 \cdot 8}{(2 \cdot 8 + 14)(8 + 14)} = \frac{56}{55}.$$

Similarly,

$$MU = \frac{b(s-a)(s-c)}{\big(2(s-c)+b\big)(s-c+b)} = \frac{15 \cdot 6 \cdot 9}{(2 \cdot 9 + 15)(9 + 15)} = \frac{45}{44},$$

whence $\frac{MU}{NV} = \frac{45}{44}\Big/\frac{56}{55} = \frac{225}{224}$, and the requested sum is $225 + 224 = 449$.

§46 Mock AIME 2019/9

For relatively prime positive integers a and b and positive real numbers c and θ, let K denote the area of the triangle with sides of length $a\sin\theta$, $b\cos\theta$, and $c\tan\theta$, given that it is positive. Suppose that if a and b remain fixed, and c and θ vary, then K achieves a maximum when $c = 85$. Find the sum of all distinct possible values of $a + b$.

Answer. 322

Let ζ be the angle opposite the side of length $c\tan\theta$. Then,

$$K = \frac{a\sin\theta \cdot b\cos\theta \cdot \sin\zeta}{2} = \frac{ab\sin 2\theta \sin\zeta}{4} \le \frac{ab}{4},$$

with equality when $\theta = 45°$ and $\zeta = 90°$. Then, by the Pythagorean Theorem,

$$\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{b}{\sqrt{2}}\right)^2 = c^2 \implies a^2 + b^2 = 2c^2.$$

Note that if $m = \frac{1}{2}(a+b)$ and $n = \frac{1}{2}(a-b)$, then

$$2c^2 = a^2 + b^2 = (m+n)^2 + (m-n)^2 = 2(m^2+n^2),$$

so $m^2 + n^2 = c^2 = 85^2$. Since $a+b$ and $a-b$ have the same parity, m and n are either both integers or both half of odd integers.

We claim that the latter case is impossible. Assume for the sake of contradiction that m and n are both half of odd integers. Let $m' = 2m$ and $n' = 2n$, so that

$$4 \cdot 85^2 = (m')^2 + (n')^2 \equiv 2 \pmod 4,$$

a contradiction. Hence, $m, n \in \mathbb{Z}$. Because $\gcd(m,n) \mid \gcd(a,b) = 1$, m and n are relatively prime, so $(m, n, 85)$ is a primitive Pythagorean triple in which $m > n$.

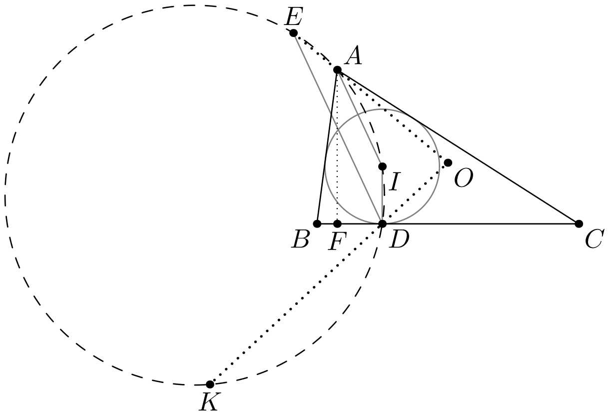
It is well known that for positive integers $p > q$, $(p^2 - q^2, 2pq, p^2 + q^2)$ generates all primitive Pythagorean triples. It is not hard to check that the only solutions to $p^2 + q^2 = 85$ are $(7, 6)$ and $(9, 2)$. These yield the Pythagorean triples $(84, 13, 85)$ and $(77, 36, 85)$. Since $2m = a + b$, the requested sum is $2(84 + 77) = 322$.

> **Remark.** The triples $(84, 13, 85)$ and $(77, 36, 85)$ yield the solutions $(a, b) = (97, 71)$ and $(113, 41)$, respectively.

§47 CIME I 2020/14

Let ABC be a triangle with sides $AB = 5$, $BC = 7$, $CA = 8$. Denote by O and I the circumcenter and incenter of $\triangle ABC$, respectively. The incircle of $\triangle ABC$ touches $\overline{BC}$ at D, and line OD intersects the circumcircle of $\triangle AID$ again at K. Then the length of DK can be expressed in the form $\frac{m\sqrt{n}}{p}$, where m, n, p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime. Find $m + n + p$.

Answer. 093



Let $\overline{AO}$ intersect (AIO) again at E, and let F be the foot from A to $\overline{BC}$. Note that $\overline{AF}$ and $\overline{AO}$ are isogonal wrt. $\angle BAC$, but $\overline{AF} \parallel \overline{ID}$. Consequently,

$$\measuredangle AID = \measuredangle IAF = \measuredangle OAI = \measuredangle EAI,$$

whence $AIDE$ is an isosceles trapezoid. In particular, $AE = ID$.

When $AB = 5$, $BC = 7$, $CA = 8$, we have $\angle A = 60^\circ$, so the area of $\triangle ABC$ is given by $K = 10\sqrt{3}$, the semiperimeter is $s = 10$, the inradius is $r = \sqrt{3}$, and the circumradius is $R = \frac{7}{\sqrt{3}}$. Plugging in the numbers,

$$OD \cdot OK = OA \cdot OE = R(R + r) = \frac{70}{3}.$$

If M denotes the midpoint of $\overline{BC}$, we can compute $BD = s - b = 2$, so $DM = \frac{3}{2}$. Since $\angle BOM = \angle A = 60^\circ$ and $BM = \frac{7}{2}$, we have $OM = \frac{7}{2\sqrt{3}}$. Thus

$$OD = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{7}{2\sqrt{3}}\right)^2} = \frac{\sqrt{57}}{3}.$$

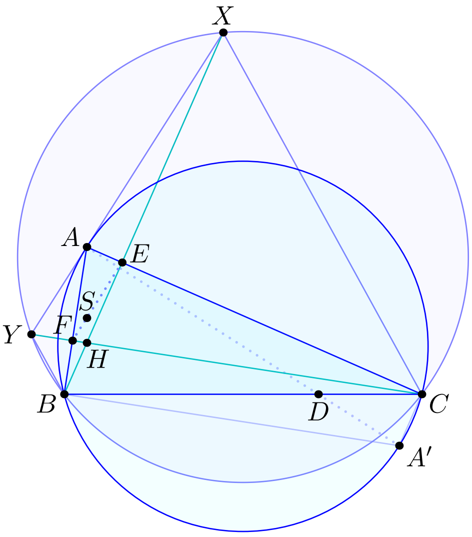
It follows that $OK = \frac{70}{\sqrt{57}}$ and $DK = \frac{17\sqrt{57}}{19}$. The requested sum is $17 + 57 + 19 = 93$.

§48 CIME I 2021/14

Let ABC be an acute triangle with circumcenter O. Points X and Y lie on the tangent to the circumcircle of $\triangle ABC$ at A so that $\overline{BX} \perp \overline{AC}$ and $\overline{CY} \perp \overline{AB}$. Line AO intersects $\overline{BC}$ at D. Suppose that $AO = 25$, $BC = 49$, and $\overline{BY} \parallel \overline{CX}$. Compute AD.

Answer. $50 - 3\sqrt{11}$

Let H be the orthocenter of $\triangle ABC$, and let E, F be the feet of the altitudes from B, C. Also let A' be the antipode of A on the circumcircle and let $S = \overline{AH} \cap \overline{EF}$.



Disregarding the condition $\overline{BY} \parallel \overline{CX}$, we contend:

> **Claim.** In general, $BCXY$ is cyclic.

Proof. Recall that $\overline{AA} \parallel \overline{EF}$, so the claim follows from Reim's theorem on $BCEF$, $BCXY$. $\square$

With $\overline{BY} \parallel \overline{CX}$, it follows that $BCXY$ is an isosceles trapezoid. In particular, $HB = HY$ and $HC = HX$. Since $\overline{SF} \parallel \overline{AY}$, we have

$$\frac{HS}{HA} = \frac{HF}{HY} = \frac{HF}{HB} = \cos A.$$

But note that $\triangle AEF \cup H \sim \triangle ABC \cup A'$, so

$$\frac{AD}{2R} = 1 - \frac{A'D}{2R} = 1 - \frac{HS}{HA} = 1 - \cos A,$$

i.e. $AD = 2R(1 - \cos A)$. We are given $R = 9$, and by the law of sines, $\sin A = \frac{49}{50}$, so $\cos A = \frac{3\sqrt{11}}{50}$, and the answer is $AD = 50 - 3\sqrt{11}$.

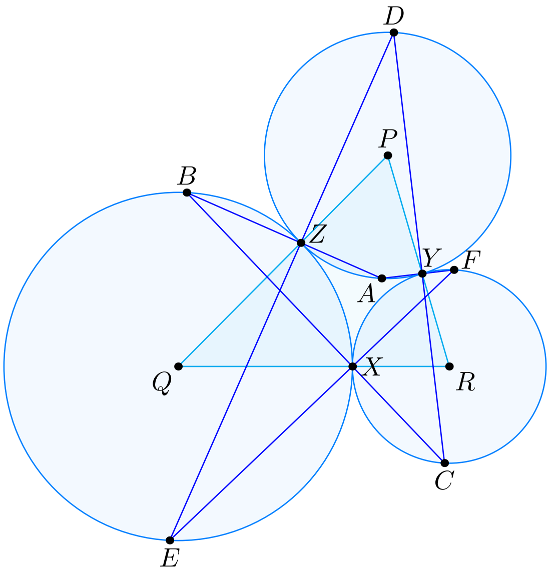
> **Remark.** For (acute) angles θ, we can show that there is an acute triangle ABC with $\angle A = \theta$ and $\overline{BY} \parallel \overline{CX}$ if and only if $\cos\theta < \frac{1}{3}$.

§49 Mock AIME 2019/13

Circles Γ_1, Γ_2, and Γ_3 are pairwise externally tangent and have diameters of length 70, 99, and 55, respectively. Suppose that Γ_2 and Γ_3 touch at X, Γ_3 and Γ_1 touch at Y, and Γ_1 and Γ_2 touch at Z. A point A is chosen on the minor arc YZ of Γ_1. Ray AZ intersects Γ_2 again at B, ray BX intersects Γ_3 again at C, ray CY intersects Γ_1 again at D, ray DZ intersects Γ_2 again at E, ray EX intersects Γ_3 again at F, and ray FY intersects Γ_1 again at G. Find the maximum possible value of $AB + BC + CD + DE + EF + FG$.

Answer. 580

Let the centers of Γ_1, Γ_2, Γ_3 be P, Q, R, respectively. By homothety, $\overline{PA}$ and $\overline{QB}$ are parallel but in opposite directions; similarly so are $\overline{QB}$, $\overline{CR}$ and $\overline{CR}$, $\overline{PD}$. Thus $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are diameters, and $G = A$.



Let $a = 70$, $b = 99$, $c = 55$ be the diameters of Γ_1, Γ_2, Γ_3, respectively, and also let $2\theta = \widehat{BX} = \widehat{CX}$. It follows that if $2\beta = \angle Q$ and $2\gamma = \angle R$, then $\widehat{AZ} = \widehat{BZ} = 2(\theta - \beta)$ and $\widehat{CY} = \widehat{DY} = 2(\theta + \gamma)$. Hence, by the Extended Law of Sines,

$$AB = AZ + BZ = (a + b)\sin(\theta - \beta).$$

Similarly, $BC = (b + c)\sin\theta$ and $CD = (c + a)\sin(\theta + \gamma)$. Remark that

$$DE = (a + b)\cos(\theta - \beta),$$

and similarly, $EF = (b + c)\cos\theta$ and $FA = (c + a)\cos(\theta + \gamma)$.

We now compute β and γ. Scale $\triangle PQR$ up by a factor of 2, so that angles are preserved. It is easy to determine that $PQ = 169$, $QR = 154$, $RP = 125$, and if S denotes the foot from P to $\overline{QR}$, $PS = 120$, $QS = 119$, $RS = 35$. Then, $\cos Q = \frac{119}{169}$ and $\cos R = \frac{7}{25}$. By the Half Angle Formulas,

$$\sin\beta = \tfrac{5}{13},\ \cos\beta = \tfrac{12}{13},\ \sin\gamma = \tfrac{3}{5},\ \cos\gamma = \tfrac{4}{5},$$

and by the Cauchy-Schwarz Inequality (after scaling back up),

$$\begin{aligned}
&AB + BC + CD + DE + EF + FA \\
&= 169\sin(\theta - \beta) + 154\sin\theta + 125\sin(\theta + \gamma) \\
&\qquad + 169\cos(\theta - \beta) + 154\cos\theta + 125\cos(\theta + \gamma) \\
&= (156\sin\theta - 65\cos\theta) + 154\sin\theta + (100\sin\theta + 75\cos\theta) \\
&\qquad + (156\cos\theta + 65\sin\theta) + 154\cos\theta + (100\cos\theta - 75\sin\theta) \\
&= 420\sin\theta + 400\cos\theta = 20(21\sin\theta + 20\cos\theta) \\
&\le 20\sqrt{(21^2 + 20^2)(\sin^2\theta + \cos^2\theta)} = 580,
\end{aligned}$$

which is easily achievable by taking the equality case of the Cauchy-Schwarz Inequality (shown in the diagram).

> **Remark.** In the above solution, we use the Cauchy-Schwarz Inequality to maximize the expression $21\sin\theta + 20\cos\theta$. Alternatively, note that if $\zeta = \tan^{-1}(\frac{20}{21})$,
>
> $$21\sin\theta + 20\cos\theta = 29(\cos\zeta\sin\theta + \sin\zeta\cos\theta) = 29\sin(\zeta + \theta) \le 29,$$

> with equality easily achievable.

> **Remark.** It appears that $\overline{AP} \perp \overline{QR}$, but this is not true. In fact, if we solve the equality case for θ, we find that $\theta = 2\tan^{-1}(\frac{3}{7})$. Hence, if S is the foot from P to $\overline{QR}$, then (oriented clockwise)
>
> $$\angle APS = 4\tan^{-1}\left(\frac{3}{7}\right) - 90^\circ \approx 2.79436205459274^\circ.$$

§50 CIME II 2020/15

Let $P_1P_2\cdots P_{72}$ be a regular dodecagon with area 1, and let $P_i = P_{i+72}$ for all integers i. Let S be the sum of the squares all positive integers $a < 72$ such that

- for all i, $P_{i-3a} \neq P_{i+a}$ and $P_{i-a} \neq P_{i+3a}$;
- for all i, lines $P_{i-3a}P_{i+a}$ and $P_{i-a}P_{i+3a}$ are not parallel, do not coincide, and intersect at a point Q_i; and
- the points Q_1, Q_2, …, Q_{72} form a polygon with positive, rational area.

Find the remainder when S is divided by 1000.

Answer. 680

Toss on the complex plane, and assume $P_0 = 1$ and the center of the polygon is 0. It suffices for OQ_i^2 to be rational. We find all angles θ such that if $\omega = e^{\theta i}$, then the intersection q of the line through ω^3 and ω^{-1} and real axis is the square root of a rational number.

Then q is the intersection of $\overline{\omega^3\omega^{-1}}$ and $\overline{(1)(-1)}$, so by the complex chord intersection formula,

$$q = \frac{\omega^3 \cdot \omega^{-1}(1 + (-1)) - 1(-1)(\omega^3 + \omega^{-1})}{\omega^3 \cdot \omega^{-1} - 1(-1)} = \frac{\omega^3 + \omega^{-1}}{\omega^2 + 1}.$$

However note that

$$\begin{aligned} \left|\omega^3 + \omega^{-1}\right|^2 &= \left[\cos(3\theta) + \cos(-\theta)\right]^2 + \left[\sin(3\theta) + \sin(-\theta)\right]^2 \\ &= 2 + 2\left[\cos(3\theta)\cos(-\theta) + \sin(3\theta)\sin(-\theta)\right] \\ &= 2 + 2\cos(4\theta), \end{aligned}$$

and similarly

$$\begin{aligned} \left|\omega^2 + 1\right|^2 &= [\cos(2\theta) + 1]^2 + \sin^2(2\theta) \\ &= 2 + 2\cos(2\theta). \end{aligned}$$

Since q is real,

$$q^2 = \frac{2 + 2\cos(4\theta)}{2 + 2\cos(2\theta)} = \frac{\cos^2 2\theta}{\cos^2\theta}$$

needs to be rational. Note that

$$|q| = \frac{\cos 2\theta}{\cos\theta} = 2\cos\theta - \frac{1}{\cos\theta} \implies 0 = 2\cos^2\theta - |q|\cos\theta - 1,$$

so since $q^2 \in \mathbb{Q}$, $\cos\theta$ is the root of a polynomial with rational coefficients and degree 4.

We turn to the following well-known lemma:

Lemma (Minimal polynomial of $\cos\theta$)
Let $\theta = \frac{2\pi k}{n}$, where k and n are relatively prime. Then the minimal polynomial of $\cos\theta$ over $\mathbb{Q}$ has degree $\frac{\varphi(n)}{2}$.

Thus the minimal polynomial of $\cos\theta$ over $\mathbb{R}$ has degree at most $\frac{\varphi(n)}{2}$, so we have $\frac{\varphi(n)}{2} \le 4$, or $\varphi(n) \le 8$.

Claim (Extracting n). If $\varphi(n) \le 8$ and $n \mid 72$, then $n \in \{2, 3, 4, 6, 8, 9, 12, 18, 24\}$.

Proof. (Details omitted.) The largest n with $\varphi(n) \le 8$ is 30, and an exhaustive check gives the above list. ☐

Now all that remains is answer extraction. We will find all $\theta < \pi$, since θ works if and only if $2\pi - \theta$ works, and $\theta = \pi$ violates the first condition.

- We first take care of $n \mid 12$, so $\theta = \frac{\pi k}{6}$ for some $0 < k < 6$. Of these, $k \in \{1, 2, 4, 5\}$ work, giving $\frac{1}{3}$, 1, 1, $\frac{1}{3}$, respectively.
- Let $n = 8$, so $\theta = \frac{\pi k}{4}$ for $k \in \{1, 3\}$. These both give zero area, thus they don't work.
- Let $n = 9$ or $n = 18$, so $\theta = \frac{\pi k}{9}$ for $k \in \{1, 2, 4, 5, 7, 8\}$. These all give irrational area.
- Let $n = 24$, so $\theta = \frac{\pi k}{24}$ for $k \in \{1, 5, 7, 11\}$. These all give irrational area.

Hence the θ that work are $\pm\frac{\pi}{6}, \pm\frac{2\pi}{6}, \pm\frac{4\pi}{6}, \pm\frac{5\pi}{6}$, which give $\frac{a}{6} \in \{1, 2, 4, 5, 7, 8, 10, 11\}$. It follows that the sum of the squares of all possible values of $\frac{a}{6}$ is

$$\sum \frac{a^2}{36} = 1^2 + 2^2 + 4^2 + 5^2 + 7^2 + 8^2 + 10^2 + 11^2 = 380,$$

from which $S = 36 \cdot 380 = 13680$, and the requested remainder is 680.

§51 Mock AIME 2018/14

Suppose 2019 chicks are sitting in a circle. Suddenly, each chick randomly pecks either the chick on its left or the chick on its right with equal probability. Let k be the number of chicks that were not pecked. The probability k is odd can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

Answer. 537

Define $p_n(k)$ as the probability k chicks were not pecked if n chicks were sitting in the circle, where n is odd.

Claim. We have

$$p_n(k) = \frac{1}{2^{n-1}}\binom{n}{2k}.$$

Proof. Number the chicks 1 to n. Suppose we have a circular string s composed of L's and R's such that a L represents a peck to the left and a R represents a peck to the right. (Note circular means that the character after the last character is the first character). A chick is not pecked iff the letter to the left is L and the letter to the right is R. Now, reorder the string s to a new circular string s' such that the chicks are in the order $1, 3, 5, \ldots, n, 2, 4, 6, \ldots, n-1$.

The number of unpecked chicks is then the number of times the string "LR" appears in s'. Note that if "LR" appears k times, "RL" also appears k times. This is because s' is circular, so there has to be a point where the string transitions from R's to L's. Note that this reasoning is similar to the concept of prefix sums. In addition, the "LR's" and "RL's" are alternating.

We can choose s' in 2^n ways. Now, we can choose the starting positions of the "LR's" and "RL's" in $\binom{n}{2k}$ ways. Note that there is no overcount because even if we choose two adjacent positions, we can construct a string such as "LRL." Consider the first position we choose in the string. It can be either "LR" or "RL," so we must multiply by 2. Note that the rest of the positions we choose rely on the previous, so they are determined by what we choose for the first. Therefore,

$$p_n(k) = \frac{2}{2^n}\binom{n}{2k} = \frac{1}{2^{n-1}}\binom{n}{2k},$$

and our claim has been proven. □

Since for $p_n(k)$ to be nonzero, $k \le \left\lfloor \frac{n}{2} \right\rfloor$, the probability k is odd is

$$\sum_{i=0}^{505} p_{2019}(2n-1) = \frac{1}{2^{2018}}\left(\sum_{i=0}^{505}\binom{2019}{4n-2}\right) = \frac{m}{2^{2018}}$$

for some m. We can evaluate m by Roots of Unity Filter. Consider the function $f(n) = (n+1)^{2019}$. Suppose $\omega = e^{\pi i/2} = i$ is a 4^{th} root of unity. Then,

$$m = \frac{f(1) - f(\omega) + f\left(\omega^2\right) - f\left(\omega^3\right)}{4} = \frac{2^{2019} - (1+i)^{2019} + 0^{2019} - (1-i)^{2019}}{4}.$$

Suppose $t = (1+i)^{2019} + (1-i)^{2019}$. Note that $m = \dfrac{2^{2019} - t}{4}$. Writing in Polar Form,

$$\begin{aligned} t &= \left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^{2019} + \left(\sqrt{2}e^{\frac{7\pi i}{4}}\right)^{2019} \\ &= 2^{\frac{2019}{2}}\left(e^{\frac{2019\pi i}{4}} + e^{\frac{2019\cdot 7\pi i}{4}}\right) \\ &= 2^{\frac{2019}{2}}\left(e^{\frac{3\pi i}{4}} + e^{\frac{5\pi i}{4}}\right) \\ &= 2^{\frac{2019}{2}}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \\ &= -2^{\frac{2019}{2}}\sqrt{2} \\ &= -2^{1010}. \end{aligned}$$

It follows that

$$m = \frac{2^{2019} + 2^{1010}}{4} = 2^{2017} + 2^{1008} = 2^{1008}\left(2^{1009} + 1\right).$$

Then, our desired probability is

$$\frac{m}{2^{2018}} = \frac{2^{1008}\left(2^{1009}+1\right)}{2^{2018}} = \frac{2^{1009}+1}{2^{1010}}.$$

It is easy to see that $\gcd\left(2^{1009}+1, 2^{1010}\right) = 1$, so we desire $2^{1009} + 1 + 2^{1010} = 3 \cdot 2^{1009} + 1$. Let this value be x. By CRT, we only seek x modulo 8 and 125.

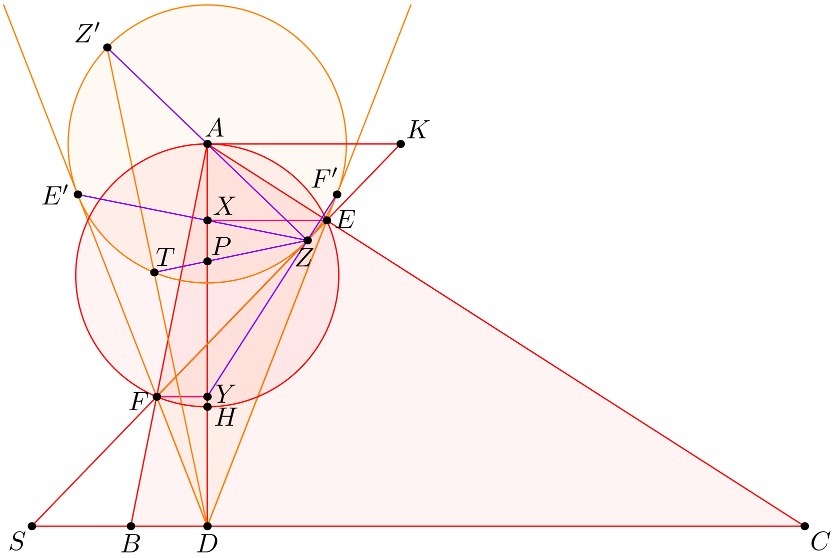
It is easy to see that $x \equiv 1 \pmod{8}$. By Euler's Formula, $2^{\phi(125)} \equiv 2^{100} \equiv 1 \pmod{125}$. Therefore, $x \equiv 3 \cdot 2^9 + 1 \equiv 3 \cdot 512 + 1 \equiv 37 \pmod{125}$. We can write $x = 8j + 1$ for some j. Then, $8j + 1 \equiv 37 \pmod{125}$, and $8j \equiv 36 \pmod{125}$, so $2j \equiv 9$ and $j \equiv \frac{9}{2} \equiv \frac{567}{126} \equiv 67 \pmod{125}$. It follows that $x \equiv 8 \cdot 67 + 1 \equiv 537 \pmod{1000}$, and we are done.

§52 Mock AIME 2019/15

Let ABC be a triangle with $AB = 11$, $BC = 19$, $CA = 20$. Let O denote its circumcenter, and let D, E, F be the feet of the altitudes from A, B, C, respectively. Points X, Y are the feet of the altitudes from E, F onto $\overline{AD}$. If $\overline{AO}$, $\overline{EF}$ intersect at Z, then there is a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. Let $\overline{ZT}$, $\overline{AD}$ intersect at P. Given that there exist relatively prime positive integers m, n such that $\frac{PX}{PY} = \frac{m}{n}$, find $m + n$.

Answer. 521

Since $\overline{BC}$, $\overline{EF}$ are antiparallel wrt. $\angle A$ and O, H are isogonal conjugates, Z is the projection of A onto $\overline{EF}$. Recall that A is the D-excenter of $\triangle DEF$, so Z is the D-extouch point. Let the D-excircle ω_D touch $\overline{DF}$, $\overline{DE}$ at E', F'.



Let Z' be the antipode of Z on ω_D. Observe that $\measuredangle DTZ = 90^\circ = \measuredangle Z'TZ$, so D, T, Z' are collinear. Since $AXZE$ is cyclic,

$$\measuredangle AZX = \measuredangle AEX = \measuredangle ACB = \measuredangle EFA = 90^\circ - \measuredangle FAZ = \measuredangle AZE',$$

so Z, X, E' are collinear. Similarly, Z, Y, F' are collinear.

> **Remark.** The collinearities $\overline{ZXE'}$ and $\overline{ZYF'}$ follow from the Iran lemma applied to excircles rather than incircles in $\triangle DEF$.

Projecting through Z,

$$-1 = (E'F'; Z'T) \overset{Z}{=} (XY; AP).$$

Finally,

$$\frac{PX}{PY} = \frac{AX}{AY} = \frac{AE \sin C}{AF \sin B} = \left(\frac{AB}{AC}\right)^2 = \frac{121}{400},$$

and the requested sum is $121 + 400 = 521$.

> **Remark.** If P' denotes the point on $\overline{EF}$ such that $\overline{BC} \parallel \overline{PP'}$, then
> $$-1 = (XY; AP) = (EF; KP').$$
> But $\overline{KA}$ is tangent to (AEF), so $\overline{AP'}$ is the A-symmedian in $\triangle AEF$; i.e. $\overline{AP'}$ bisects $\overline{BC}$.

§53 OMO Spring 2020/28

Let A_0BC_0D be a convex quadrilateral inscribed in a circle ω. For all integers $i \geq 0$, let P_i be the intersection of $\overline{A_iB}$ and $\overline{C_iD}$, let Q_i be the intersection of $\overline{A_iD}$ and $\overline{BC_i}$, let M_i be the midpoint of $\overline{P_iQ_i}$, and let $\overline{M_iA_i}$ and $\overline{M_iC_i}$ intersect ω again at A_{i+1} and C_{i+1}, respectively. The circumcircles of $\triangle A_3M_3C_3$ and $\triangle A_4M_4C_4$ intersect at two points U and V.

If $A_0B = 3$, $BC_0 = 4$, $C_0D = 6$, $DA_0 = 7$, then UV can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c such that $\gcd(a, c) = 1$ and b is squarefree. Find $100a + 10b + c$.

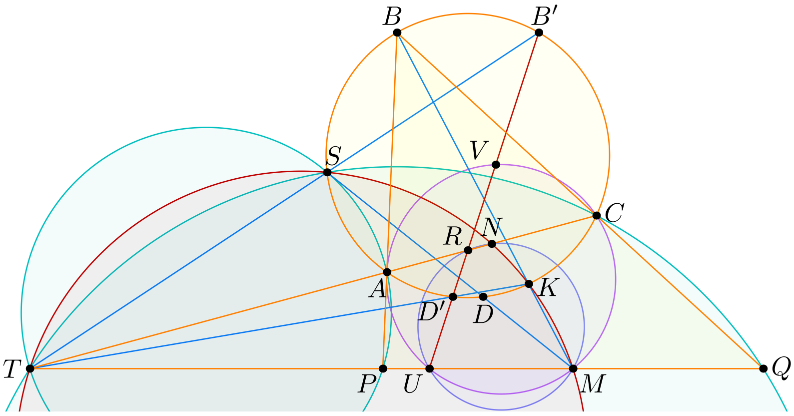
Answer. 4375

In fact the circles $(A_iM_iC_i)$ are coaxial for all nonnegative integers i. Let $R = \overline{A_0C_0} \cap \overline{BD}$. Since $\overline{A_iA_{i+1}} \cap \overline{C_iC_{i+1}} = M_i$ for all i, by Brokard's theorem $\overline{A_iC_i} \cap \overline{A_{i+1}C_{i+1}}$ lies on the polar of M_i. Hence we have by induction that all $\overline{A_iC_i}$ concur on at R. Thus $\overline{P_iQ_i}$ is always the polar of R, so the points P_i and Q_i are all collinear on a line ℓ.

Lemma

Let $ABCD$ be a quadrilateral with circumcircle ω and circumcenter O, and let $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{AD} \cap \overline{BC}$, $R = \overline{AC} \cap \overline{BD}$. The reflection of $\overline{BD}$ in $\overline{OR}$ intersects $\overline{PQ}$ at U, and M is the midpoint of $\overline{PQ}$. Let V be the foot from O to $\overline{RU}$. Then $CMUAV$ is cyclic.

Proof. Let $T = \overline{AC} \cap \overline{PQ}$. Let S be the Miquel point of $APQC$ and K the Miquel point of $AQPC$. Then the spiral similarity at S sending $\overline{AC}$ to $\overline{PQ}$ sends N to M, and the spiral similarity at K sending $\overline{AC}$ to $\overline{QP}$ sends N to M. Hence S, K, T, M, N are concyclic.



Let B' and D' be the reflections of B and D aross $\overline{OR}$. Note that $\measuredangle STP = \measuredangle SAP = \measuredangle SAB = \measuredangle SB'B$, so T, S, B' collinear. Similarly T, K, D' collinear. By Reim's theorem on ω, (MNT), we have B, K, M collinear and D, S, M collinear.

Hence $\measuredangle RNM = \measuredangle TNM = \measuredangle TSM = \measuredangle B'SD = \measuredangle D'B'B = \measuredangle RUM$, thus $RNMU$ is cyclic. Since $-1 = (AC; RT)$, we have $TA \cdot TC = TR \cdot TN = TU \cdot TM$, so $CMUA$ is cyclic.

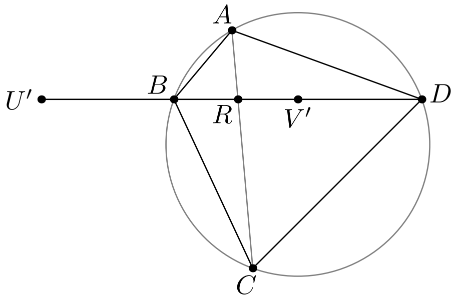
Finally V is the midpoint of $\overline{B'D'}$. Since $-1 = (B'D'; RU)$, we have $RU \cdot RV = RB' \cdot RD' = RA \cdot RC$, as desired. $\square$

Let $A = A_0$ and $C = C_0$. In summary, to construct U and V, we let O be the circumcenter, $R = \overline{AC} \cap \overline{BD}$, and let B_0 and D_0 be the reflections of B and D across $\overline{OR}$. Then U is the intersection of $\overline{B_0D_0}$ and the polar of R, and V is the midpoint of $\overline{B_0D_0}$.

Reflect U and V across $\overline{OR}$, so that U' is the intersection of $\overline{BD}$ and the polar of R, and V' is the midpoint of $\overline{BD}$. The task is to compute $U'V'$.

However $-1 = (BD; RU')$, so $RB \cdot RD = RU' \cdot RV'$. Thus

$$U'V' = RV' + \frac{RB \cdot RD}{RV'} = RV' + \frac{(BD/2)^2 - RV'^2}{RV'} = \frac{BD^2}{4RV'}.$$



The rest is just a routine computation. Let $x = \cos \angle BAD$. By Law of Cosines on $\triangle ABD$ and $\triangle BCD$, we have BD^2 is equal to the two quantities

$$\begin{aligned} 3^2 + 7^2 - 2 \cdot 3 \cdot 7x &= 4^2 + 6^2 + 2 \cdot 4 \cdot 6x \\ \implies 58 - 42x &= 52 + 48x \implies x = 1/15. \end{aligned}$$

Hence $BD^2 = 276/5$. Now

$$\frac{BR}{RD} = \frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle ADC)} = \frac{AB \cdot BC}{AD \cdot DC} = \frac{2}{7},$$

so $BR/BD = 2/9$ and $RV'/BD = 5/18$. Thus

$$U'V' = \frac{BD}{4 \cdot RV'/BD} = \frac{9}{10} BD = \frac{9\sqrt{345}}{25},$$

and the requested sum is $900 + 3450 + 25 = 4375$.

§54 CJMO 2020/1

Let N be a positive integer, and let S be the set of all tuples with positive integer elements and a sum of N. For instance, $t_1 = (N)$, $t_2 = (1, 1, N-2)$, $t_3 = (1, N-1)$, and $t_4 = (N-1, 1)$ are all distinct tuples in S. For all tuples t, let $p(t)$ denote the product of all the elements of t. For instance, $p(t_1) = N$, $p(t_2) = N - 2$, and $p(t_3) = p(t_4) = N - 1$.

Evaluate the expression (where we sum over all elements t of S)

$$\sum_{t \in S} p(t).$$

Let $F_0 = 0$, $F_1 = 1$, and for all $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$. The answer is F_{2N}.

First solution To show this, we use strong induction. The base case, $N = 1$, is clear. Let $f(N)$ be the answer for N. It can be seen that if the hypothesis holds for all integers less than k, then by picking the first element of the tuple first, $f(k)$ is equal to

$$f(k) = \sum_{i=0}^{k-1}(k-i)F_{2i} = \sum_{i=0}^{k-1}\sum_{j=0}^{i} F_{2j} = \sum_{i=0}^{k-1} F_{2i+1} = F_{2k},$$

and the induction is complete.

Second solution For all positive integers i, let S_i denote the subset of S that contains all tuples with cardinality i.

Claim. For all i,

$$\sum_{t\in S_i} p(t) = \binom{N-1+i}{2i-1}.$$

First proof by combinatorial argument. The desired sum is bijective with splitting up a line of N items into i sections, and picking a representative from each section. Using the Stars and Bars method, we can add in $i-1$ dividers. We can pick $2i-1$ items, each of which is either a representative or divider. Since between two representatives there is exactly one divider, which of these selected items is a divider follows. Hence, there are $\binom{N-1+i}{2i-1}$ ways to pick sections and representatives, as desired. □

Second proof by strong induction. Let $S_i(k)$ be the number of tuples with cardinality i whose elements sum to k. It suffices to show that

$$\sum_{t\in S_i(k)} p(t) = \binom{k-1+i}{2i-1}.$$

The base case, $i = 1$, is trivial. Then, we can pick the first element of each tuple first, so by the Hockey Stick Identity,

$$\begin{aligned}\sum_{t\in S_i(k+1)} p(t) &= \sum_{j=1}^{k-i+2}\left(j \sum_{t\in S_i(k+1-j)} p(t)\right) = \sum_{j=1}^{k-i+2}\left(j\binom{k+i-j}{2i-1}\right)\\ &= \sum_{\ell=2i-1}^{k+i-1}\sum_{j=2i-1}^{\ell}\binom{j}{2i-1} = \sum_{\ell=2i-1}^{k+i-1}\binom{\ell+1}{2i} = \binom{k+1+i}{2i+1},\end{aligned}$$

as required. □

We then have that

$$\sum_{t\in S} p(t) = \sum_{i=1}^{N}\sum_{t\in S_i} p(t) = \sum_{i=1}^{N}\binom{N-1+i}{2i-1} = \sum_{i=1}^{N}\binom{N-1+i}{N-i} = F_{2N},$$

as desired.

§55 CJMO 2021/1

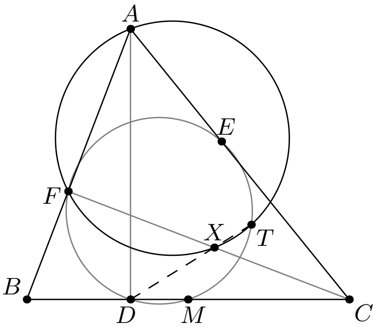
Let ABC be an acute triangle, and let the feet of the altitudes from A, B, C to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ be D, E, F, respectively. Points X and Y lie on lines CF and BE respectively such that $\angle XAD = \angle DAB$ and $\angle YAD = \angle DAC$. Prove that X, D, Y are collinear.

We present three solutions.

First solution, by nine-point circle Let $\overline{DX}$ intersect the nine-point circle again at T. Then

$$\measuredangle FAX = 2\measuredangle BAD = 2\measuredangle FBM = \measuredangle FMB = \measuredangle FTD = \measuredangle FTX,$$

so $AFXT$ is cyclic, and $\angle ATD = \angle ATX = 90^\circ$. Point T is unique and symmetric, so analogously T lies on $\overline{DY}$, and X and Y both lie on line DT, as desired.



Second solution, by Menelaus Let B', C' be the reflections of B, C across D, so that $X = \overline{AB'} \cap \overline{CF}$, $Y = \overline{AC'} \cap \overline{BE}$.

In what follows, lengths are directed. By Menelaus on $\triangle ADB'$, $\triangle ADC'$ with traversals $\overline{HXC}$, $\overline{HYB}$,

$$\frac{AX}{XB'} = -\frac{CD}{B'C} \cdot \frac{HA}{DH} \quad \text{and} \quad \frac{C'Y}{YA} = -\frac{BC'}{DB} \cdot \frac{HD}{AH}.$$

Finally

$$\frac{AX}{XB'} \cdot \frac{B'D}{DC'} \cdot \frac{C'Y}{YA} = \left(-\frac{CD}{B'C} \cdot \frac{HA}{DH}\right)\left(\frac{B'D}{DC'}\right)\left(-\frac{BC'}{DB} \cdot \frac{HD}{AH}\right) = -1,$$

where we use $B'D = DB$, $DC' = CD$, $B'C = -BC'$. The collinearity follows from Menelaus on $\triangle AB'C'$.

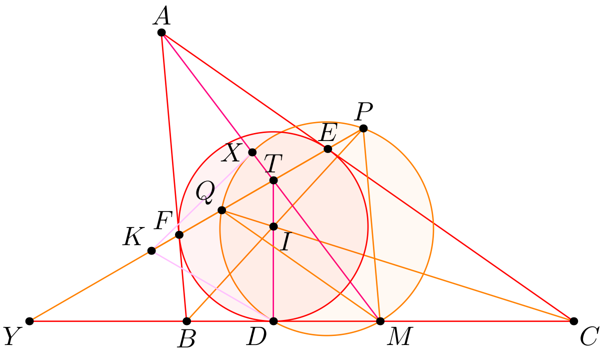
Third solution, by Desargue involution Let H be the orthocenter and let $D' = \overline{BC} \cap \overline{XY}$. By DDIT from A to $BCXY$, we obtain the involutive pairs $A(BX; CY; HD')$. Projecting onto $\overline{BC}$, we obtain the involutive pairs $(BX; CY; DD')$. From the first two pairs, this involution is reflection across D; hence $D = D'$, as desired.

§56 CJMO 2019/3

Let I be the incenter of $\triangle ABC$, and M be the midpoint of $\overline{BC}$. Let Ω be the nine-point circle of $\triangle BIC$. Suppose that $\overline{BC}$ intersects Ω at a point $D \neq M$. If Y is the intersection of $\overline{BC}$ and the A-intouch chord, and X is the projection of Y onto $\overline{AM}$, prove that X lies on Ω, and the intersection of the tangents to Ω at D and X lies on the A-intouch chord of $\triangle ABC$.

(The nine-point circle of $\triangle ABC$ is the circumcircle of its medial triangle, and if the incircle touches $\overline{AC}$ and $\overline{AB}$ at E and F, respectively, then $\overline{EF}$ is the A-intouch chord.)

First solution, by harmonic bundles Let the incircle of $\triangle ABC$ touch $\overline{AC}$ and $\overline{AB}$ at E and F, respectively. Furthermore, let $P = \overline{BI} \cap \overline{EF}$ and $Q = \overline{CI} \cap \overline{EF}$. By the Iran Lemma, $\angle BPC = \angle BQC = 90^\circ$, so $MP = MQ$. Let $T = \overline{AM} \cap \overline{EF}$. Obviously the incircle of $\triangle ABC$ touches $\overline{BC}$ at D.



It is well-known that T lies on $\overline{ID}$. Then, by Ceva-Menelaus,

$$-1 = (B,C;D,Y) \overset{I}{=} (P,Q;T,Y).$$

However, by construction, $\angle TXY = 90^\circ$, so by a well-known lemma, $\overline{XT}$ bisects $\angle PXQ$. Since $\triangle DPQ$ is the orthic triangle of $\triangle BIC$, $(DPQ) = \Omega$. However, because $MP = MQ$, M is the midpoint of $\widehat{PQ}$ in Ω. By Apollonian circles, X is unique point on $\overline{AM}$ such that $\overline{XM}$ bisects $\angle PXQ$, whence $X \in (PMQ)$. Then, notice that

$$-1 = (P,Q;T,Y) \overset{M}{=} (P,Q;X,D),$$

and it follows that the intersection of the tangents to Ω at D and X lies on $\overline{PQ}$, which is the A-intouch chord, as required.

Second solution, by angle chasing Assume WLOG $\angle B > \angle C$. Clearly D is the point where the incircle touches $\overline{BC}$. Let $\overline{EF}$ be the A-intouch chord, H be the orthocenter of $\triangle BIC$, and N and S be the midpoints of $\overline{HI}$ and $\overline{HC}$, respectively. It is well-known that $\overline{AM}, \overline{EF}, \overline{ID}$ concur at a point, say T. Since $TXYD$ is cyclic,

$$\angle MXD = \angle TXD = \angle TYD = 180 - \angle CEY - \angle YCE = 90 - \frac{A}{2} - C.$$

However, if H_I and I_A denote the reflections of H over D and M, respectively, so that they lie on the circumcircle of $\triangle BIC$. If L is the intersection of the angle bisector of $\angle BIC$ with (BIC), since $\widehat{H_IL} = \widehat{LI_A}$,

$$\angle MND = \angle I_AIH_I = 2\angle LIH_I = 2\angle CID - 2\angle LIC = 90 - \frac{A}{2} - C,$$

and $X \in \Omega$. If K is the midpoint of $\overline{YT}$ so that K is the circumcenter of $YDTX$, then

$$\measuredangle KXD = 90 - \measuredangle DYX = 90 - \measuredangle DTX = 90 - \measuredangle DTM = \measuredangle TMD = \measuredangle XMD,$$

so $\overline{KX}$ is tangent to Ω. Furthermore, $KD^2 = KX^2$, so we are done.

§57 CAMO 2020/1

Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (meaning f takes positive real numbers to positive real numbers) be a nonconstant function such that for any positive real numbers x and y,

$$f(x)f(y)f(x+y) = f(x) + f(y) - f(x+y).$$

Prove that there is a constant $a > 1$ such that

$$f(x) = \frac{a^x - 1}{a^x + 1}$$

for all positive real numbers x.

Rewrite our functional equation as

$$f(x+y)=\frac{f(x)+f(y)}{1+f(x)f(y)}.$$

The key claim is that $f(x)<1$ or $f\equiv 1$.

Claim 1. $f(x)\ge 1 \implies f(\frac{x}{2})=1$.

Proof. Plugging in $(\frac{x}{2},\frac{x}{2})$ into our functional equation gives

$$\frac{2f(\frac{x}{2})}{1+f(\frac{x}{2})^2}=f(x)\ge 1 \implies \left(f\left(\frac{x}{2}\right)-1\right)^2\le 0 \implies f\left(\frac{x}{2}\right)=1,$$

as desired. □

Claim 2. $f(x)\ge 1 \implies f\equiv 1$.

Proof. By Claim 1, there exists y such that $f(y)=1$. Furthermore, if $f(y)=1$ then $f(\frac{y}{2})=1$ by Claim 1, so we can take y infinitely small. Then, by our functional equation,

$$f(x+y)=\frac{f(x)+1}{1+f(x)}=1$$

for all x, so $f\equiv 1$. □

Now, discard the trivial solution $f\equiv 1$. We have that $f(x)<1$ for all x. Let

$$g(x)=\ln\left(\frac{1+f(x)}{1-f(x)}\right).$$

Then,

$$g(x+y)=\ln\left(\frac{(1+f(x))(1+f(y))}{(1-f(x))(1-f(y))}\right)=g(x)+g(y),$$

so g satisfies Cauchy's Functional Equation. Since $0<f(x)<1$ for each x, we have $g(x)>0$, so g is bounded and there exists a positive constant k such that $g(x)\equiv kx$. Thus $a=e^k$, and we are done.

§58 c851151h1975686p13709352

Prove that for every integer k, there is a permutation $(a_1,a_2,\dots,a_{2010})$ of the numbers $(1,2,\dots,2010)$ such that

$$a_1a_2+a_2a_3+a_3a_4+\cdots+a_{2009}a_{2010}+a_{2010}a_1\equiv k \pmod{2011}.$$

Let $p=2011$, and let L denote the left-hand expression.

Proof for k a nonzero quadratic residue: Let $a_i=ti^{-1}$ for all i. Then

$$L\equiv t^2\left[\frac{1}{p-1}+\sum_{x=1}^{p-1}\frac{1}{x(x+1)}\right]\equiv t^2\left[\frac{1}{p-1}+\sum_{x=1}^{p-1}\left(\frac{1}{x}-\frac{1}{x+1}\right)\right]\equiv t^2 \pmod{p}.$$

Proof for k a nonzero quadratic nonresidue: Since $p \equiv 3 \pmod 4$, the quadratic nonresidues modulo p are exactly the negatives of the quadratic residues. Let $a_i = ti$ for all i. Then

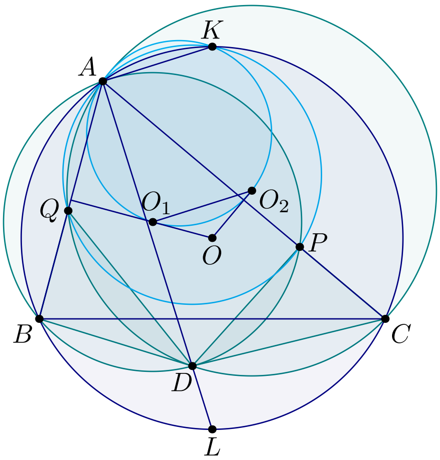
$$L \equiv t^2 \left[p - 1 + \sum_{x=1}^{p-1} x(x+1)\right] \equiv t^2 \left(-1 + \sum_{x=1}^{p} x + \sum_{x=1}^{p} x^2\right) \equiv -t^2 \pmod p.$$

Proof for $k \equiv 0$: Let g be a primitive root modulo p, and take $a_i = g^i$ for all i. Then

$$L \equiv \sum_{i=0}^{p-2} g^{2i+1} = \frac{g\left(1 - g^{2(p-1)}\right)}{1 - g^2} \equiv 0 \pmod p.$$

§59 CJMO 2020/5

Let ABC be a triangle, and D be a point on the internal angle bisector of $\angle BAC$ but not on the circumcircle of $\triangle ABC$. Suppose that the circumcircle of $\triangle ABD$ intersects $\overline{AC}$ again at P and the circumcircle of $\triangle ACD$ intersects $\overline{AB}$ again at Q. Denote by O_1 and O_2 the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Prove that the circumcenters of $\triangle ABC$, $\triangle APQ$, and $\triangle AO_1O_2$ are collinear.



Let K be the midpoint of arc BAC on (ABC). I claim that (APQ) and (AO_1O_2) pass through K, from which the result is obvious.

Claim 1. K lies on (APQ).

Proof. By construction, D is the center of spiral similarity sending $\overline{BQ}$ to $\overline{PC}$. However, since $\overline{AD}$ bisects $\angle BAP$, $DB = DP$, so $\triangle DBQ \cong \triangle DPC$, and $BQ = PC$. Since $\measuredangle QBK = \measuredangle ABK = \measuredangle ACK = \measuredangle PCK$, by SAS, $\triangle KBQ \cong \triangle KCP$, so K is the Miquel point of $BCPQ$, and K lies on (APQ), as desired. ☐

Claim 2. $OO_1 = OO_2$, where O is the circumcenter of $\triangle ABC$.

Proof. Note that $\overline{O_1O_2}, \overline{OO_1}, \overline{OO_2}$ are the perpendicular bisectors of $\overline{AD}, \overline{AB}, \overline{AC}$, respectively, so

$$\measuredangle(\overline{OO_1}, \overline{O_1O_2}) = \measuredangle(\overline{AB}, \overline{AD}) = \measuredangle(\overline{AD}, \overline{AC}) = \measuredangle(\overline{O_1O_2}, \overline{OO_2}),$$

as required. ☐

Claim 3. K lies on (AO_1O_2).

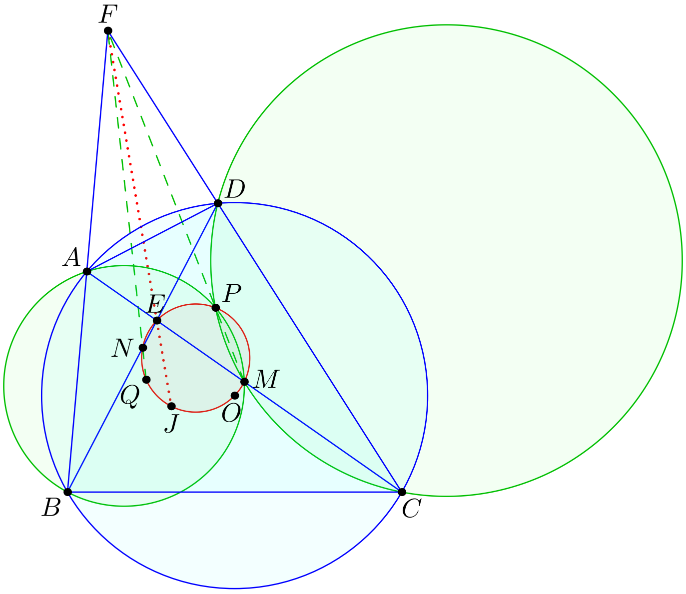
Proof. Since $\overline{O_1O_2}$ and $\overline{AK}$ are both perpendicular to $\overline{AD}$, and O lies on both of their perpendicular bisectors, AO_1O_2K must be an isosceles trapezoid, so it is cyclic. $\square$

Hence, (ABC), (APQ), and (AO_1O_2) are coaxial, so their centers are collinear, as desired.

§60 CJMO 2021/5

Let $ABCD$ be a cyclic quadrilateral, and let M and N be the midpoints of $\overline{AC}$ and $\overline{BD}$. The circumcircles of $\triangle ABM$ and $\triangle CDM$ intersect again at P; the circumcircles of $\triangle ABN$ and $\triangle CDN$ intersect again at Q; the circumcircles of $\triangle ADM$ and $\triangle BCM$ intersect again at R; and the circumcircles of $\triangle ADN$ and $\triangle BCN$ intersect again at S. Prove that $\overline{PQ} \parallel \overline{RS}$.

Let O be the circumcenter and let $E = \overline{AC} \cap \overline{BD}$, $F = \overline{AB} \cap \overline{CD}$. In fact, we will show $\overline{OE} \perp \overline{PQ}$. By symmetry the desired result follows from here.



First O, M, N, E are concyclic; say their circumcircle is ω. I contend:

Claim. P lies on $\overline{FM}$ and ω. Analogously Q lies on $\overline{FN}$ and ω.

Proof. To see $P \in \overline{FM}$, apply Radical Axis theorem on (ABM), (CDM), $(ABCD)$.

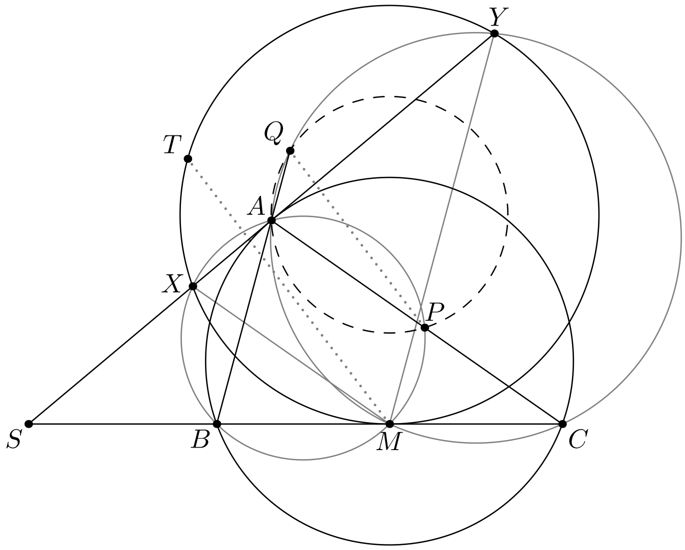
Let $J = (ABE) \cap (CDE) \setminus E$ be the Miquel point of $ABDC$. By properties of the Miquel point, J is the foot from O to $\overline{EF}$, whence $J \in \omega$. Then $FM \cdot FP = FA \cdot FB = FE \cdot FJ$, so $P \in \omega$. $\square$

Finally $\measuredangle PME = \measuredangle FMA = -\measuredangle FND = -\measuredangle QNE$ by $\triangle FAC \sim \triangle FDB$, so $\widehat{EP} = \widehat{EQ}$ on ω. Since $\overline{OE}$ is a diameter of ω, we have $\overline{OE} \perp \overline{PQ}$, and we are done.

§61 CAMO 2021/4

Let ABC be a triangle and let M be the midpoint of $\overline{BC}$. The circumcircle of $\triangle ABM$ intersects $\overline{AC}$ again at P, and the circumcircle of $\triangle ACM$ intersects $\overline{AB}$ again at Q. Select point T on the circumcircle of $\triangle MPQ$ such that $\overline{MT} \parallel \overline{PQ}$, and let ω be the circle through T tangent to $\overline{BC}$ at M. The circumcircles of $\triangle ABM$ and $\triangle ACM$ intersect ω again at X and Y. Prove that line XY is tangent to the circumcircle of $\triangle ABC$.

Redefine X and Y as the second intersections of the tangent to (ABC) at A with (ABM) and (ACM) respectively. The task is to show that M, X, Y, T lie on a circle tangent to $\overline{BC}$.



To see that (MXY) is tangent to $\overline{BC}$, let $S = \overline{AA} \cap \overline{BC}$. From here we have

$$\begin{aligned} SA^2 &= SB \cdot SC \\ SA \cdot SX &= SM \cdot SB \\ SA \cdot SY &= SM \cdot SC. \end{aligned}$$

Combining these, $SX \cdot SY = SM^2$, as required.

To show that T also lies on (MXY), it suffices to show the center of (MXY) lies on the perpendicular bisector of $\overline{PQ}$. In fact, I claim that the centers of (MXY) and (APQ) coincide. Note that

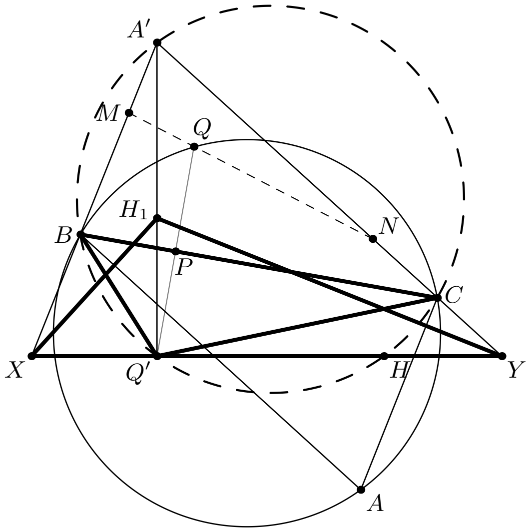
$$\measuredangle XAP = \measuredangle XAC = \measuredangle ABC = \measuredangle ABM = \measuredangle APM$$

and $\measuredangle YAQ = \measuredangle AQM$, so $APMX$ and $AQMY$ are isosceles trapezoids. It follows that the perpendicular bisectors of $\overline{MX}$ and $\overline{AP}$ conincide, and the perpendicular bisectors of $\overline{MY}$ and $\overline{AQ}$ coincide. Thus (MXY) and (APQ) are concentric, as desired.

§62 CAMO 2020/4

Let ABC be a triangle and Q a point on its circumcircle. Let E and F be the reflections of Q over $\overline{AB}$ and $\overline{AC}$, respectively. Select points X and Y on line EF such that $\overline{BX} \parallel \overline{AC}$ and $\overline{CY} \parallel \overline{AB}$, and let M and N be the reflections of X and Y over B and C respectively. Prove that M, Q, N are collinear.

First solution, by spiral similarity Let A' be the point such that $ABA'C$ is a parallelogram, so $X \in \overline{A'B}$ and $Y \in \overline{A'C}$. Define H as the orthocenter of $\triangle ABC$, H_1 as the orthocenter of $\triangle A'XY$, Q' as the reflection of Q over $\overline{BC}$, and P as the foot of Q on $\overline{BC}$.



To begin, observe that Q' must lie on $\overline{XY}$ by homothety on a Simson line. This, in conjunction with $\angle BQC = \angle BHC$ and the well-known fact that H lies on $\overline{XY}$ implies that Q' lies on $(A'BHC)$, so it must be the foot of A' on $\overline{XY}$.

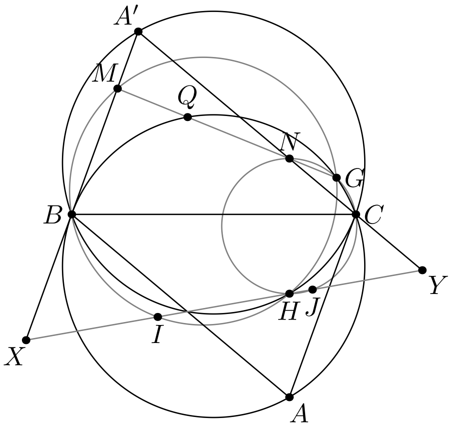
Next, we have

$$\measuredangle H_1XY = 90^\circ - \measuredangle XYA = \measuredangle CA'Q' = -\measuredangle Q'BC,$$

and similarly $\measuredangle XYH_1 = -\measuredangle BCQ'$; therefore, $\triangle Q'BC \sim \triangle H_1XY$ and, as a consequence, degenerate triangles PBC and $Q'XY$ are also similar. Collinearity of M, Q, N follows from the mean geometry theorem.

Second solution, by angle chasing Let A' be the point such that $ABA'C$ is a parallelogram, so $X \in \overline{A'B}$ and $Y \in \overline{A'C}$, and let O and H be the circumcenter and orthocenter of $\triangle ABC$. Since $\overline{XY}$ is the image of the Simson line from Q under homothety $(Q, 2)$[21], we know H lies on $\overline{XY}$.

Let I and J be the projections of M and N onto $\overline{XY}$, so that B is the center of (MXI) and C is the center of (NYJ). Then $HIBM$ and $HJCN$ are cyclic with diameters $\overline{HM}$ and $\overline{HN}$; say they intersect again at G.



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[21] sometimes called the *Steiner line*

Then $\angle MGH = \angle NGH = 90^\circ$, so $G \in \overline{MN}$. Furthermore

$$\measuredangle BGM = \measuredangle IGB = \measuredangle IHB \quad \text{and} \quad \measuredangle NGC = \measuredangle CHJ.$$

Adding these, $\measuredangle BGC = \measuredangle CHB = \measuredangle BAC$, so G lies on (ABC).

Say $\overline{MN}$ intersects (ABC) again at Q' and $\overline{XY}$ intersects $(A'BC)$ again at D. Then recalling that $\measuredangle BGM = \measuredangle IHB$,

$$\measuredangle BCQ' = \measuredangle BGQ' = \measuredangle BGM = \measuredangle IHB = \measuredangle DHB = \measuredangle DCB,$$

and similarly $\measuredangle Q'BC = \measuredangle CBD$, so Q' and D are reflections across $\overline{BC}$, and $\overline{XY}$ is the Steiner line of Q'.

Third solution, by length Let H be the orthocenter of $\triangle ABC$ and D the reflection of Q over $\overline{BC}$. Since $\overline{XY}$ is the image of the Simson line from Q under homothety $(Q, 2)$, we know H and D lie on $\overline{XY}$.

Let U and V lie on $\overline{XY}$ such that $\overline{BU}$ and $\overline{CV}$ are perpendicular to $\overline{BC}$.

Claim. $DU : DV = DX : DY$.

Proof. Let D' be the foot from Q to $\overline{BC}$ (i.e. the midpoint of $\overline{QD}$). Remark that $\overline{A'H}$ is a diameter of $(A'BC)$ by orthocenter reflections, so D is the foot from A' to $\overline{XY}$. Note that

$$\frac{DU}{DV} = \frac{D'B}{D'C} = \frac{DB}{DC} \cdot \frac{\cos \angle DBC}{\cos \angle DCB} = \frac{DB}{DC} \cdot \frac{\cos \angle DA'C}{\cos \angle DA'B} = \frac{DB}{DC} \cdot \frac{\sin \angle A'YD}{\sin \angle A'XD},$$

but

$$\frac{DX}{DY} = \frac{A'X}{A'Y} \cdot \frac{\sin \angle BA'D}{\sin \angle CA'D} = \frac{A'X}{A'Y} \cdot \frac{DB}{DC} = \frac{DB}{DC} \cdot \frac{\sin \angle A'YD}{\sin \angle A'XD},$$

as claimed. □

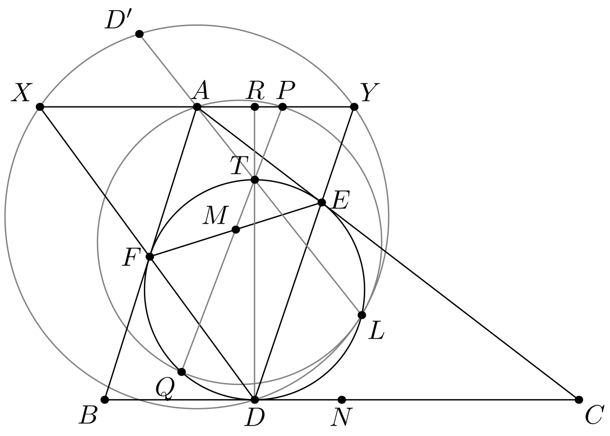
Let X' and Y' be the reflections of X and Y over $\overline{BC}$. We have

$$\frac{MX'}{NY'} = \frac{UX}{VY} = \frac{DX}{DY} = \frac{QX'}{QY'},$$

so $\overline{X'Y'} \cap \overline{MN}$ is the reflection of D across $\overline{BC}$, which is Q. This completes the proof.

§63 CAMO 2020/3

Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Point M is the midpoint of $\overline{EF}$, and T is the point on ω such that $\overline{DT}$ is a diameter. Line MT meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines DF and DE intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.



Let $\overline{DT}$ intersect $\overline{AP}$ at R, and let $\overline{AT}$ intersect ω again at L.

Claim 1. T lies on $\overline{EX}$ and $\overline{FY}$, T is the orthocenter of $\triangle DXY$, and A is the midpoint of $\overline{XY}$.

Proof. Redefine $X = \overline{DF} \cap \overline{TE}$ and $Y = \overline{DE} \cap \overline{TF}$. Since $\angle DET = \angle DFT = 90^\circ$, T is the orthocenter of $\triangle DXY$. Thus, $\overline{DT} \perp \overline{XY}$, so $\overline{XY} \parallel \overline{BC}$.

By the Three Tangents lemma, the tangents to ω at E and F intersect at the midpoint of $\overline{XY}$; but this is A, thus recovering the original definitions of X and Y. ☐

Claim 2. L lies on (DXY) and (APQ).

Proof. Since A is the midpoint of $\overline{XY}$ and T is the orthocenter of $\triangle DXY$, $\overline{AT}$ passes through D', the antipode of D on (DXY). Note that $\angle DLD' = \angle DLT = 90^\circ$, so L lies on (DXY).

Now $L \in (APQ)$ follows from $TA \cdot TL = TR \cdot TD = TP \cdot TQ$, thus proving the claim. ☐

Finally, since $\overline{TM}$ is the T-symmedian of $\triangle TXY$ and L is the Miquel point of $XYEF$,

$$\frac{LX}{LY} = \frac{XF}{YE} = \frac{TX}{TY} = \left(\frac{PX}{PY}\right)^2.$$

It follows that $\overline{LP}$ is a symmedian of $\triangle LXY$. Since $\overline{LA}$ and $\overline{LP}$ are isogonal, (LAP) and (DXY) are tangent, and we are done.

§64 CAMO 2020/5

Let $f(x) = x^2 - 2$. Prove that for all positive integers n, the polynomial

$$P(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)\dots)) - x$$

can be factored into two polynomials with integer coefficients and equal degree.

First solution, by irreducibility We first prove a lemma.

Lemma

Let P be a monic polynomial. If P^2 has integer coefficients, then so does P.

Proof. Suppose there is a polynomial without this property, and henceforth let P be such a polynomial of minimal degree. Note $P^2 \in \mathbb{Z}[x]$, so let the factorization of P^2 into factors that are powers of irreducible polynomials in $\mathbb{Z}[x]$ be P_1, P_2, …, P_k.

By construction they do not share roots with one another. Since they multiply to the square of a polynomial in $\mathbb{R}[x]$, they are all squares in $\mathbb{R}[x]$. If $k > 1$ they must all be in $\mathbb{Z}[x]$ by the minimality of the degree of P. Hence $k = 1$ and $P(x)^2 = Q(x)^r$ for some r. If r is even we are done, so assume r is odd. Then $Q(x)$ must be the square of a polynomial in $\mathbb{R}[x]$, but it is irreducible, contradiction. ☐

Consider the sequence defined by $y_n = \frac{1}{2}f^n(2x)$. For $n > 0$,

$$y_n = \tfrac{1}{2}(f^{n-1}(2x)^2 - 2) = \tfrac{1}{2}(4y_{n-1}^2 - 2) = 2y_{n-1}^2 - 1.$$

If $|y_0| < 1$, say that $y_0 = \cos\theta$ for some angle θ. It follows that $y_n = \cos(2^n\theta)$ for all n, whence solutions to $P(x) = 0$ obey $\cos(2^n\theta) = \cos\theta$. Thus the set of solutions to $P(x) = 0$ includes

$$2\cos\left(\frac{2\pi k}{2^n - 1}\right) \quad \text{and} \quad 2\cos\left(\frac{2\pi k}{2^n + 1}\right) \quad \text{for all } k.$$

The former describes 2^{n-1} distinct roots and the latter describes $2^{n-1} + 1$ distinct roots. The only root they share is 1, so we have described all 2^n solutions.

> **Claim.** Let m be an odd integer. The monic polynomial with roots $2\cos(\frac{2\pi k}{m})$, $0 \le k < m$, has integer coefficients.

Proof. Let $g_n(2\cos\theta) = 2\cos(n\theta)$. The key observation is that

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x).$$

Indeed, this rewrites to

$$2\cos\theta\cos(n-1)\theta = \cos n\theta + \cos(n-2)\theta,$$

which is just the product-to-sum identity. With this, g_n is a monic integer polynomial of degree n for all n, but the polynomial $g_m(x)$ is exactly the polynomial we need. ☐

The polynomial described by the above claim is precisely the square of the polynomial with roots $2\cos(\frac{2\pi k}{m})$, $0 \le k < \frac{m}{2}$, whence it has integer coefficients by the lemma. Let Q be this integer polynomial for $m = 2^n + 1$ and R for $m = 2^n - 1$.

Clearly P is monic. We can factor out $x - 2$ from Q and add it to R (thus 2 is a double root), thereby giving two factors of P with integer coefficients and equal degree.

Second solution, by polynomial transformation (Raymond Feng, unedited) We are trying to show for an odd integer m that

$$\prod_{k=0}^{\frac{m-1}{2}} \left(x - 2\cos\left(\frac{2\pi k}{m}\right)\right)$$

is an integer polynomial. Note that

$$\prod_{k=1}^{m-1} \left(x - e^{\frac{2\pi k}{m}}\right) = \frac{x^m - 1}{x - 1},$$

which has integer coefficients. However, we can also write this as

$$\begin{aligned}
\frac{x^m - 1}{x - 1} &= \prod_{k=1}^{\frac{m-1}{2}} \left(\left(x - e^{\frac{2\pi k}{m}}\right)\left(x - e^{\frac{2\pi(m-k)}{m}}\right)\right) \\
&= \prod_{k=1}^{\frac{m-1}{2}} \left(x^2 + 1 - 2\cos\left(\frac{2\pi k}{m}\right)x\right) \\
&= x^{\frac{m-1}{2}} \cdot \prod_{k=1}^{\frac{m-1}{2}} \left(x + \frac{1}{x} - 2\cos\left(\frac{2\pi k}{m}\right)\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}\prod_{k=1}^{\frac{m-1}{2}}\left(x+\frac{1}{x}-2\cos\left(\frac{2\pi k}{m}\right)\right) &= \frac{1}{x^{\frac{m-1}{2}}}\frac{x^m-1}{x-1}\\ &= \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} x^k\\ &= 1+\sum_{k=1}^{\frac{m-1}{2}}\left(x^k+\frac{1}{x^k}\right).\end{aligned}$$

The final expression is an integer polynomial in $x+\frac{1}{x}$ (since all expressions of the form $x^k+\frac{1}{x^k}$ are expressible as integer polynomials of $x+\frac{1}{x}$), thus,

$$\prod_{k=1}^{\frac{m-1}{2}}\left(x+\frac{1}{x}-2\cos\left(\frac{2\pi k}{m}\right)\right)$$

is an integer polynomial in $x+\frac{1}{x}$. This implies that

$$\prod_{k=0}^{\frac{m-1}{2}}\left(x-2\cos\left(\frac{2\pi k}{m}\right)\right)=(x-2)\cdot\prod_{k=1}^{\frac{m-1}{2}}\left(x-2\cos\left(\frac{2\pi k}{m}\right)\right)$$

is an integer polynomial in x, as desired. Then finish as in TheUltimate123's solution.

Third solution, by explicit factorization (Andrew Gu, unedited) Note that $f\left(t+\frac{1}{t}\right)=t^2+\frac{1}{t^2}$, so

$$\begin{aligned}P\left(t+\frac{1}{t}\right) &= t^{2^n}+\frac{1}{t^{2^n}}-t-\frac{1}{t}\\ &= \frac{(t^{2^n+1}-1)(t^{2^n-1}-1)}{t^{2^n}}\\ &= \frac{(t^{2^n}+t^{2^n-1}+\cdots+1)(t^{2^n}-t^{2^n-1}-t+1)}{t^{2^n}}\\ &= \left(t^{2^{n-1}}+t^{2^{n-1}-1}+\cdots+\frac{1}{t^{2^{n-1}-1}}+\frac{1}{t^{2^n}}\right)\left(t^{2^{n-1}}-t^{2^{n-1}-1}-\frac{1}{t^{2^{n-1}-1}}+\frac{1}{t^{2^{n-1}}}\right).\end{aligned}$$

Now let A, B be the polynomials such that

$$\begin{aligned}A\left(t+\frac{1}{t}\right) &= t^{2^{n-1}}+t^{2^{n-1}-1}+\cdots+\frac{1}{t^{2^{n-1}-1}}+\frac{1}{t^{2^n}}\\ B\left(t+\frac{1}{t}\right) &= t^{2^{n-1}}-t^{2^{n-1}-1}-\frac{1}{t^{2^{n-1}-1}}+\frac{1}{t^{2^{n-1}}}.\end{aligned}$$

We can check that $P(x)=A(x)B(x)$ is the desired factorization.

§65 CAMO 2021/5

Do there exist positive integers m and n such that

$$\frac{n^2-1}{m^2-n^2-1}$$

is also a positive integer?

No such m and n exist. Let $m > n > 2$, and assume for contradiction the expression evaluates to an integer k. Set $x = m + n$ and $y = m - n$, so that

$$k = \frac{(x-y)^2 - 4}{(x+y)^2 - (x-y)^2 - 4} = \frac{x^2 - 2xy + y^2 - 4}{4xy - 4}.$$

This rearranges to

$$0 = x^2 - (2y + 4yk)x + \left(y^2 + 4k - 4\right),$$

so any solution (x, y) with $x > y$ (without loss of generality) generates another solution

$$(x, y) \mapsto (2y + 4yk - x, y) = \left(\frac{y^2 + 4x - 4}{x}, y\right).$$

Consider the minimal solution (x, y) with $x, y > 0$. Evidently $x \neq y$, so (since x and y have equal parity), $x - y \geq 2$. Since (x, y) is minimal,

$$\begin{aligned} \frac{y^2 + 4k - 4}{x} &\geq x \\ \implies y^2 + 4(k-1) &\geq x^2 \\ \implies y^2 + \frac{x^2 - 6xy + y^2}{xy - 1} &\geq x^2 \\ \implies (xy - 1)(x^2 - y^2) &\leq x^2 - 6xy + y^2 \\ \implies x^2y - y^3 - x &\leq x - 6y \\ \implies y \cdot x^2 - 2x - \left(y^3 + 6y\right) &\leq 0. \end{aligned}$$

This is a quadratic polynomial P in x. Since its vertex is $\frac{1}{y} < 1$, and $x \geq y + 2$, we know $P(x) \leq P(y + 2) \leq 0$, hence

$$\begin{aligned} 0 &\geq y(y+2)^2 - 2(y+2) - \left(y^3 + 6y\right) \\ &= 4\left(y^2 - y - 1\right), \end{aligned}$$

which fails whenever $y \geq 2$.

Thus $y = 1$, and

$$\frac{(x-1)^2 - 4}{4(x-1)} \in \mathbb{Z}$$

Since $\frac{(x-1)^2-4}{x-1} \in \mathbb{Z}$, we have $x - 1 \mid 4$. A finite-case check shows that only $x = 3$ works, but this implies $k = 0$.

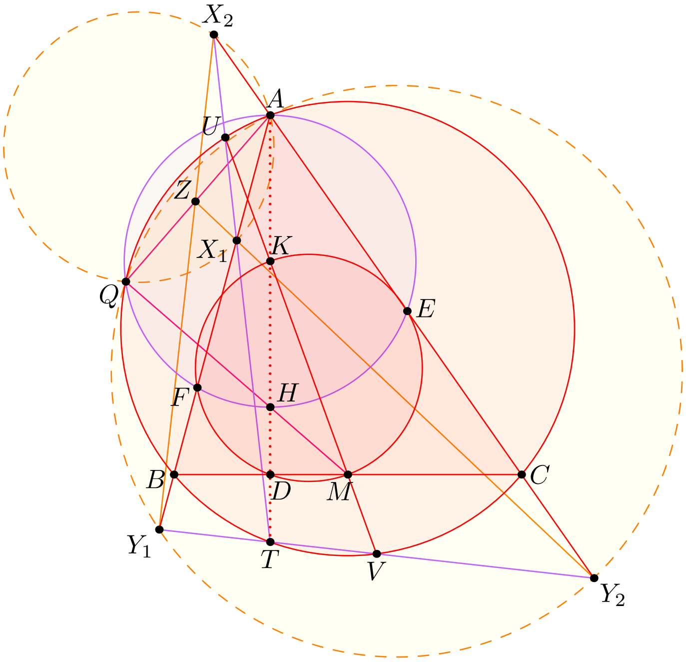
§66 CAMO 2021/3

Let ABC be an scalene triangle with circumcircle Γ and orthocenter H, and let K and M be the midpoints of $\overline{AH}$ and $\overline{BC}$, respectively. Line AH intersects Γ again at T, and line KM intersects Γ at U and V. Lines TU and TV intersect lines AB and AC at X and Y, respectively, and point W lies on line KM such that $\overline{AW} \perp \overline{HM}$. If Z is the reflection of A over W, prove that X, Y, Z are collinear.

By symmetry in U, V, the clear interpretation of the problem is as follows:

> Let ABC be a triangle with circumcircle Γ, orthocenter H, and let K, M be the midpoints of $\overline{AH}$, $\overline{BC}$. Line KM intersects Γ at U, V. Let $\overline{TU}$ intersect $\overline{AB}$, $\overline{AC}$ at X_1, X_2 and let $\overline{TV}$ intersect $\overline{AB}$, $\overline{AC}$ at Y_1, Y_2. Finally, $Z = \overline{X_1Y_2} \cap \overline{X_2Y_1}$. Prove that (i) $\overline{AZ} \perp \overline{HM}$ and that (ii) line KM bisects $\overline{AZ}$.

In what follows, DEF is the orthic triangle, and Q lies on Γ so that $\angle AQH = 90°$. It follows that Q lies on $\overline{HM}$ and (AEF).



Proof of (i) Since $\overline{AQ} \perp \overline{HM}$, it suffices to prove A, Z, Q collinear.

Recall that $-1 = (BC;TQ)$, say by noting that

$$\frac{QB}{QC} = \frac{BF}{CE} = \frac{BH}{CH} = \frac{BT}{CT}.$$

By Ceva-Menelaus in $\triangle X_2Y_1Y_2$, we have

$$-1 = (Y_1, Y_2; T, \overline{AZ} \cap \overline{Y_1Y_2}) \overset{A}{=} (B, C; T, \overline{AZ} \cap \Gamma),$$

as needed.

Proof of (ii) Observe that Q is the spiral center between $\overline{BC}$, $\overline{EF}$. The key is the following claim:

> **Claim.** A spiral similarity sends $\overline{BC} \cup M$ to $\overline{X_1X_2} \cup U$.

Proof. Consider the (unique) points X_1', X_2' on $\overline{AB}$, $\overline{AC}$ such that $U \in \overline{X_1'X_2'}$ and a spiral similarity at Q sends $\overline{BC}$ to $\overline{X_1'X_2'}$. By gliding principle, the midpoints of $\overline{BC}$, $\overline{EF}$, $\overline{X_1'X_2'}$ are collinear; i.e. U is the midpoint of $\overline{X_1'X_2'}$.

It will suffice to show $X_1 = X_1'$ and $X_2 = X_2'$, i.e. $T \in \overline{X_1'X_2'}$. Recall that $BMEQ$ is cyclic by properties of the Miquel point, so

$$\measuredangle QUX_1' = \measuredangle QMB = \measuredangle QEB = \measuredangle QEH = \measuredangle QAH = \measuredangle QAT = \measuredangle QUT,$$

and the claim follows. □

Finally, U is the midpoint of $\overline{X_1X_2}$ and V is the midpoint of $\overline{Y_1Y_2}$, so $\overline{UV}$ is the Gauss line of $X_1Y_1X_2Y_2$. It passes through the midpoint of $\overline{AZ}$, and we are done.

§67 CAMO 2020/6

Let n be a positive integer. Eric and a squid play a turn-based game on an infinite grid of unit squares. Eric's goal is to capture the squid by moving onto the same square as it.

Initially, all the squares are colored white. The squid begins on an arbitrary square in the grid, and Eric chooses a different square to start on. On the squid's turn, it performs the following action exactly 2020 times: it chooses an adjacent unit square that is white, moves onto it, and sprays the previous unit square either black or gray. Once the squid has performed this action 2020 times, all squares colored gray are automatically colored white again, and the squid's turn ends. Moreover, the squid is claustrophobic, so at no point in time is it ever surrounded by a closed loop of black or gray squares. On Eric's turn, he performs the following action at most n times: he chooses an adjacent unit square that is white and moves onto it. Note that the squid can trap Eric in a closed region, so that Eric can never win.

Eric wins if he ever occupies the same square as the squid. Suppose the squid has the first turn, and both Eric and the squid play optimally. Both Eric and the squid always know each other's location and the colors of all the squares. Find all positive integers n such that Eric can win in finitely many moves.

Let $s = 2020$. In general, the answer for $s \geq 8$ is $n \geq 2s - 5$. Henceforth, by "distance," we refer to the length of the shortest path between Eric and the squid that does not intersect the squid ink. For all shown diagrams, a white circle represents Eric's initial position, a black circle represents the squid's initial position, a gray line represents Eric's path, and a solid line represents the squid's path.

Proof of upper bound: Say $n < 2s - 5$. The key here is that Eric cannot get close enough to the squid, or the squid can surround Eric. We use the following estimate.

> **Claim 1.** Suppose it's the squid's turn, the squid has only used gray ink (so there are no black squares), and the distance between Eric and the squid is $d \leq s - 6$. Then the squid wins.

Proof. Consider the following picture.

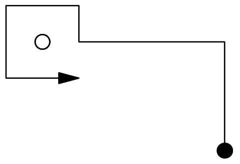


Figure 1: Surrounding Eric

Here it takes the squid $d - 1$ moves to get to the closest point adjacent to Eric, and then 7 moves to surround him. Hence if $s \geq d + 6$, the squid can surround Eric using black ink, as claimed. ☐

Now assume that at an arbitrary point in time, Eric is a distance of $d \geq s - 5$ away, and it's the squid's turn. As the squid continues moving, it is able to increase its distance from Eric by s, so after the squid's turn, Eric may be as many as $s + d = 2s - 5$ units away. Thus if $n < 2s - 5$, Eric is unable to capture the squid.

Proof of lower bound: Since Eric moves at most n times every turn, it suffices to show Eric can win when $n = 2s - 5$. In fact I claim Eric can win on his first move.

Without loss of generality the squid starts at $(0, 0)$. We claim Eric can win on his first turn if he starts from $(-1, -s + 6)$. Call the final position of the squid the *destination*. Assume that the squid only places black ink. This is worse for Eric, and does not affect the squid's first turn.

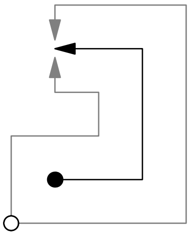


Figure 2: Eric's strategy

First note that the squid cannot surround Eric, as that would take at least $s+1$ moves. Furthermore the squid cannot surround itself, as then it can only perform finitely many moves before losing.

We consider two paths, the "right" and "left" paths, from Eric to the destination; in summary, Eric reaches some point on the squid's path, and moves around the path in the two possible directions until it reaches the destination. We will show at least one of these two paths has length at most $n = 2s - 5$. Let T be the sum of the lengths of the two paths.

> **Claim 2.** If the squid's ink ever blocks Eric from following the ink, then Eric is able to "jump" to another point on the squid's ink that is closer to the destination, and this decreases the length of Eric's path.

Proof. Without loss of generality, Eric is on the right-hand path. Say a *blockade* is when some part of the ink blocks the square directly to the right of some square on the path. There are two possible blockades, as shown below.

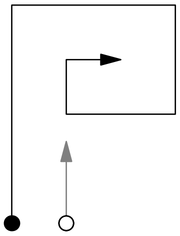


Figure 3: Left-hand blockade

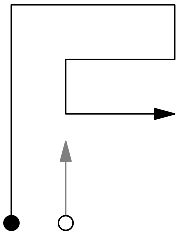


Figure 4: Right-hand blockade

The blockade must be closer to the destination than the current position. Otherwise, we will have already dealt with the intersection before.

In the case of a left-hand blockade, the squid has trapped itself in a closed region, and there are only finitely many squares it can reach. Thus the squid will eventually lose, contradiction. In the case of a right-hand blockade, once Eric reaches the blockade, he can turn right and skip a portion of the blockade. Thus this is in Eric's favor. ☐

We proceed with the computation. Eric's first step is to reach a point adjacent to some point on the squid's path. (It is possible that the destination is the only possible point Eric can reach in this manner.)

It takes $s-5$ moves to reach a point adjacent to the closest point on the ink path. Once there, we split off into two different directions to surround the squid's path. As shown in Figure 2 and Figure 5, the union of these two paths (with the first $s-5$ moves omitted) forms a cycle, and $T' = T - 2(s-5)$ is the length of this cycle.

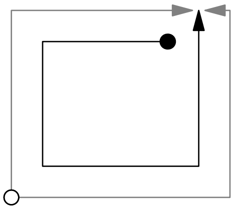


Figure 5: Surrounding the squid ink

Claim 3. $T' \leq 2(s+1)$.

Proof. Note that for each corner flanked by two sides of the cycle, the averages of the additional lengths the two surrounding paths gain equals the total length the corner contributes to the length of the squid's ink. Refer to the bottom-right corner in Figure 2.

Thus each unit along the ink corresponds to at most two units along the length of T'. Finally we need to consider the final move from a point adjacent to the destination onto the destination. This yields an additional 2 units, so $T' \leq 2(s+1)$, as claimed. ☐

By definition, we have $T \leq 2(2s-4)$. Note that if $T < 2(2s-4)$, by Pigeonhole, one of Eric's two choices has length at most $n = 2s-5$, so Eric can win. Assume instead that equality holds.

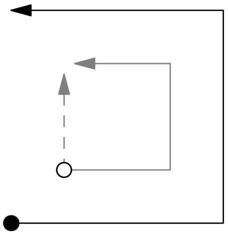


Figure 6: Skipping the U-turn

If there are any *U-turns*, as shown above, Eric can shorten one of the paths. Then the sum of his two choices is now less than $2(2s-4)$, so we may finish as above. Henceforth also assume there are no U-turns. Thus in the worst-case scenario, all of the squid's moves are either north or east.

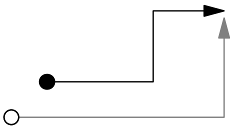


Figure 7: The equality case

If the last move of the squid's turn is to the east, Eric can move all the way to the east, and then up. An analogous argument holds if the last move is to the north. The length of this path is equal to the taxicab distance from Eric to the destination, which is $(s-5)+s = 2s-5 = n$, as desired.

Finally, the answer is $n \geq 2s-5 = 4035$, and we are done.