Density

and Complexity

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§1 Complexity: review

§1.1 Definition

Using **complexity**, we consider the order of magnitude of some function. An intuitive example is that any polynomial of degree d has complexity x^d and $x \log_{2020} x + 2019$ has complexity $x \log x$.

It is easy to think of complexity as the "term of largest degree," but a more rigorous definition is as follows: we'll say $a \sim b$ (where a, b are functions in x) whenever there are positive constants C_1 and C_2 such that

$$C_1 \le \frac{a}{b} \le C_2$$

for sufficiently large x.

For convenience, we'll use **big-O notation**; let $a \sim b$ be equivalent to $a = O(b)$. There are some more annoying details with big-O notation, but we all know what you mean.

> **Remark.** $a = O(b)$ actually means a/b is bounded above but not necessarily below. To be accurate, we would use Θ instead, but nobody cares. (To complete the set, Ω means bounded below.)

§1.2 Example

Example 1.1 (USAMO 1995/4)

Suppose q_0, q_1, q_2, ... is an infinite sequence of integers satisfying the following two conditions:

(i) $m - n$ divides $q_m - q_n$ for $m > n \ge 0$, and

(ii) there is a polynomial P such that $|q_n| < P(n)$ for all n.

Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

Walkthrough. Intuitively, Q should have the same degree as P, so let's define Q as the rational polynomial with $Q(i) = q_i$ for $i = 0, \dots, d$.

(a) First convince yourself Q is unique and exists; moreover $\deg Q \le d$. For convenience, scale up the entire problem so that Q has integer coefficients.

(b) Verify that for all n,

$$q_n \equiv Q(n) \pmod{\operatorname{lcm}(n, \dots, n-d)}.$$

(c) Prove that

$$\operatorname{lcm}(n, \dots, n-d) \sim n^{d+1},$$

and use that to show $q_n = Q(n)$ for sufficiently large n.

(d) Conclude that $q_n = Q(n)$ for all n.

§2 Density bounding

§2.1 The main idea

Suppose you wanted to find a number n such that satisfies a bunch of properties simultaneously; the main idea for this handout is to look at how many n fail to satisfy the property for a certain prime p, then use the union bound and sum over all p to get an upper bound on the number of failing p.

Thus, the main idea is **double counting** and then applying **union bound** to prove the existence of infinitely many n satisfying some property we want.

We will be routinely using the following theorems (which can also be cited on olympiads):

Theorem 2.1 (Prime Number Theorem)

Let $\pi(n)$ be the number of primes at most n. Then

$$\pi(n) \sim \frac{n}{\log n}$$

Theorem 2.2 (Strong form of Dirichlet)

For $\gcd(c, d) = 1$ let $\pi_d(n, c)$ be the number of primes congruent to c modulo d less than n. Then

$$\pi_d(n, c) \sim \frac{n}{\phi(d) \log n}.$$

§2.2 A motivating example

Example 2.3 (Ukraine TST 2007/12)

Prove that there are infinitely many positive integers n for which all the prime divisors of $n^2 + n + 1$ are not more then $\sqrt{n}$.

Walkthrough. We will show there are infinitely many n for which the prime divisors of $n^8 + n^4 + 1$ are no more than n^2. The reason why is this:

(a) Verify that

$$n^8 + n^4 + 1 = \left(n^4 - n^2 + 1\right)\left(n^2 - n + 1\right)\left(n^2 + n + 1\right).$$

(b) Impose the condition that $n \equiv 1 \pmod 3$, and verify all prime factors of $\left(n^2 - n + 1\right)\left(n^2 + n + 1\right)$ are less than n^2. Then it will suffice to show infinitude of $n \equiv 1 \pmod 3$ such that all prime factors of $n^4 - n^2 + 1$ are less than n^2.

(c) When does there exist $p \geq n^2$ with $p \mid n^4 - n^2 + 1$? Use double-counting to count how many n fail for some p.

(d) Take the union bound to upper-bound the number of failing $n \in \{1, 4, 7, \ldots, 3N - 2\}$ by

$$\sum_{p<3N} 4 \sim \frac{N}{\log N}.$$

(e) Verify that the density of failing n is strictly less than 1.

Remark. Note the really lax use of constant factors; it's possible to show that the number of solutions to

$$n^4 - n^2 + 1 \equiv 0 \pmod p$$

for $p \geq n^2$ is at most 1, but this extra constant factor doesn't matter since our 2 densities differ by

> a factor of $\log N$. In general, constant factors can essentially be ignored.

Interestingly, this problem can be solved by considering cyclotomic polynomials, which the interested reader can attempt on their own (it is not related to the density topics covered in the handout).

§2.3 Useful bounds

- (Prime number theorem) $\pi(n) \sim \frac{n}{\log n}$.
- $\sum_{n\geq 1} \frac{1}{n^s}$ diverges for $s < 1$, converges for $s > 1$.
  - $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} < \frac{5}{3}$.
- (Sums over primes) $\sum_p \frac{1}{p}$ diverges
  - $\sum_p \frac{1}{p^2} < \frac{1}{2}$.

Exercise 2.4. Verify by hand (using $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$) that

$$\sum_p \frac{1}{p^2} < \frac{1}{2}.$$

§3 Walkthroughs

Example 3.1 (China Southeast 2020/11.7)

Arrange all square-free positive integers in ascending order $a_1, a_2, \ldots$. Prove that there are infinitely many positive integers n such that $a_{n+1} - a_n = 2020$.

Walkthrough. We want to show the infinitude of n where n, $n+2020$ are squarefree but $n+1$, $\ldots$, $n+2019$ are divisible by squares.

(a) Impose a restriction of the form $n \equiv A \pmod B$ such that for all such n, none of $n+1$, $\ldots$, $n+2019$ are squarefree. We will then show that for infinitely many such n, both n and $n+2020$ are squarefree.

(b) Analyze the condition "$q^2 \mid n$ or $q^2 \mid n+2020$." (This is the condition we want to fail.)

- **(i)** Use double counting to determine the density of $n \equiv A \pmod B$ with $q^2 \mid n$ for each fixed q.
- **(ii)** Use double counting to determine the density of $n \equiv A \pmod B$ with $q^2 \mid n+2020$ for each fixed q.
- **(iii)** Use the union bound to upper-bound the density of $n \equiv A \pmod B$ with $q^2 \mid n$ or $q^2 \mid n+2020$.

(c) Use the union bound to sum over q, upper-bounding the density of $n \equiv A \pmod B$ for which there exists q dividing either n or $n+2020$.

(d) The above density should be less than 1, so conclude.

Example 3.2 (USAMO 2014/6)

Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \dots, n\}$, then

$$\min\{a, b\} > (cn)^{n/2}.$$

Walkthrough. To simplify computation, we only use $i, j \in \{1, \dots, n\}$.

(a) Consider a $n \times n$ grid defined by the points $(a+i, b+j)$. Place the least prime factor of $\gcd(a+i, b_j)$ in each cell. Fix a prime p and provide an upper bound for the number of cells that contain a p.

(b) The goal is to show many of the primes in the grid are large. Prove that at least $n^2/2$ cells contain primes greater than n. (To do this, use the union bound to upper-bound cells with primes at most n.)

(c) Verify by Pigeonhole that some row $a+i$ contains $n/2$ primes greater than n; moreover, verify all these primes are distinct.

(d) Show that

$$a + i > n^{n/2};$$

conclude.

§4 Additional practice

Problem 4.1 (strong USAMO 2014/6)**.** Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \dots, n\}$, then

$$\min\{a, b\} > (cn)^n.$$

Problem 4.2 (Iran 2001/3/2)**.** Determine whether there exists a sequence a_1, a_2, a_3, ... of nonnegative reals such that

$$a_n + a_{2n} + a_{3n} + \cdots = \frac{1}{n}$$

for every integer n.

Problem 4.3 (Canada 2020/4)**.** Let $S = \{4, 8, 9, 16, \dots\}$ be the set of perfect powers. Prove that if we arrange the elements of S into an increasing sequence $a_1 < a_2 < \cdots$, then there are infinitely many n for which $9999 \mid a_{n+1} - a_n$.

Problem 4.4 (ISL 2015 N6)**.** Consider a function $f : \mathbb{N} \to \mathbb{N}$ with the following two properties:

(i) if $m, n \in \mathbb{N}$, then $\frac{f^n(m) - m}{n} \in \mathbb{N}$; and

(ii) the set $\mathbb{N} \setminus \{f(n) : n \in \mathbb{N}\}$ is finite.

Prove that the sequence $f(1) - 1$, $f(2) - 2$, $f(3) - 3$, ... is periodic.

Problem 4.5 (China TST 2018/1/5)**.** Given a positive integer k, call n *good* if among

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$$

at least $0.99n$ of them are divisible by k. Show that exists some positive integer N such that among 1, 2, ..., N, there are at least $0.99N$ good numbers.

Problem 4.6 (InfinityDots 2019/5)**.** Is there a finite nonempty set S of points in the plane which forms at least $|S|^2$ cyclic harmonic quadrilaterals?

Problem 4.7 (ZhiHu)**.** Prove there exists a sequence of 2020 consecutive integers such that no two have the same number of divisors. (For example, 27, 28, 29, 30 satisfy the condition for 4 consecutive integers.)

§5 Solutions to walkthroughs

§5.1 Solution 1.1 (USAMO 1995/4)

Let d be the degree of P, and let Q be the (unique) rational polynomial with $Q(i) = q_i$ for $i = 0, \dots, d$. Appropriately scale up everything so that Q has integer coefficients.

Lemma

For all d,

$$\operatorname{lcm}(n, n-1, \dots, n-d) = O\big(n^{d+1}\big).$$

Proof. Evidently $\operatorname{lcm}(n, n-1, \dots, n-d) \le n(n-1)\cdots(n-d) = O\big(n^{d+1}\big)$. The lower bound can be attained easily via basically any bound; for instance,

$$\operatorname{lcm}(n, \dots, n-d) \ge \frac{n(n-1)\cdots(n-d)}{\prod_{i<j} \gcd(n-i, n-j)} \ge \frac{n(n-1)\cdots(n-d)}{\prod_{i<j}(j-i)} = O\big(n^{d+1}\big).$$

□

Claim. For sufficiently large n, we have $q_n = Q(n)$.

Proof. We must have $q_n \equiv q_i = Q(i) \equiv Q(n) \pmod{n-i}$ for $i = 0, \dots, d$, so

$$q_n \equiv Q(n) \pmod{\operatorname{lcm}(n, \dots, n-d)}.$$

If $q_n \ne Q(n)$, then for some nonzero k we have $q_n = Q(n) + k \operatorname{lcm}(n, \dots, n-d) = O\big(n^{d+1}\big)$, which will exceed $P(n)$ in absolute value for sufficiently large n. □

To finish the proof, observe that for every m, we have $q_m \equiv q_n = Q(n) \equiv Q(m) \pmod{n-m}$ for sufficiently large n. By taking n large, $q_m = Q(m)$, as needed.

§5.2 Solution 2.3 (Ukraine TST 2007/12j)

Observe that

$$n^{4040} + n^{2020} + 1 = \left(n^2 - n + 1\right)\left(n^2 + n + 1\right)\left(\frac{n^{2020} - n^{1010} + 1}{n^2 - n + 1}\right)\left(\frac{n^{2020} + n^{1010} + 1}{n^2 + n + 1}\right).$$

Each term is less than n^{2020} for large n, so we are done.

§5.3 Solution 3.1 (China Southeast 2020/11.7)

To begin, let p_1, p_2, ..., p_{2019} be primes, with $P = p_1 p_2 \cdots p_{2019}$, and let c be an integer (that exists by Chinese Remainder theorem) such that $c \equiv -i \pmod{p_i^2}$ for $i = 1, \dots, 2019$. Then for each k of the form $k = P^2 i + c$, the integers $k+1$, ..., $k+2019$ are all divisible by squares.

It will suffice to show there are infinitely many k in the arithmetic sequence $i \mapsto P^2 i + c$ such that k and $k + 2020$ are both squarefree.

Remark. In what follows, we will cite the fact that $\sum_p \frac{1}{p^2} < \frac{1}{2}$, where the sum ranges over primes p. Here is an easy proof: first note that

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \sum_n \frac{1}{n^2} - \sum_{n \text{ even}} \frac{1}{n^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} < \frac{5}{4}$$

by $\pi^2 < 10$. Then
$$\sum_p \frac{1}{p^2} < -\frac{1}{1^2} + \frac{1}{2^2} + \sum_{n \text{ odd}} \frac{1}{n^2} < \frac{1}{2}.$$

Observe that for fixed primes q, the density of positive integers i for which $q^2 \mid P^2 i + c$ is $\le 1/q^2$. Similarly i with $q^2 \mid P^2 i + c + 2020$ have density $\le 1/q^2$, so the density of i for which q^2 divides either $P^2 i + c$ or $P^2 i + c + 2020$ is $\le 2/q^2$.

But the density of i for which there exists q that divide either $P^2 i + c$ or $P^2 i + c + 2020$ is at most
$$\sum_q \frac{2}{q^2} < 1,$$
so there are infinitely many i for which $P^2 i + c$ and $P^2 i + c + 2020$ are squarefree, as desired.

§5.4 Solution 3.2 (USAMO 2014/6)

To simplify computation, we only use $i, j \in \{1, \dots, n\}$. We will prove the stronger bound $\min\{a, b\} > (cn)^n$ for sufficiently large n.

Let $\varepsilon = 10^{-10}$ be small. Consider the $n \times n$ grid defined by the points $(a+i, b+j)$ where $i, j \in \{1, \dots, n\}$, and in each cell place the least prime factor of $\gcd(a+i, b+j)$. Note that each prime p divides at most $(1 + n/p)^2$ cells of the grid.

Claim. For n large, at most $n^2/2$ of the cells of the grid contain a prime $p < \varepsilon n^2$.

Proof. The number of primes covered is
$$\sum_{p < \varepsilon n^2} \left(1 + \frac{n}{p}\right) \le \pi(\varepsilon n^2) + 2n \sum_{p < \varepsilon n^2} \frac{1}{n} + n^2 \sum_{p<n} \frac{1}{p^2} < \frac{n^2}{2}$$
for sufficiently large n. □

Remark. Some more details on the estimates: where $s = \varepsilon n^2$, we have
$$\pi(s) = \frac{s}{\log s}\left(1 + O\left(\frac{1}{\log s}\right)\right) = o\left(n^2\right)$$
$$\sum_{p<s} \frac{1}{p} < \sum_{k=1}^{s} \frac{1}{p} = O(\log s) = o(n^2)$$
$$\sum_{p<s} \frac{1}{p^2} < \sum_p \frac{1}{p^2} \approx 0.452 < \frac{1}{2}.$$
For curiosity sake, the best bound for the second expression is $\sum_{p<s} \frac{1}{p} = \log\log s \cdot (1 + o(1))$.

Remark. Here is an easy proof of $\sum_p \frac{1}{p^2} < \frac{1}{2}$. Note that
$$\sum_{n \text{ odd}} \frac{1}{n^2} = \sum_n \frac{1}{n^2} - \sum_{n \text{ even}} \frac{1}{n^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} < \frac{5}{4}$$
by $\pi^2 < 10$. Then
$$\sum_p \frac{1}{p^2} < -\frac{1}{1^2} + \frac{1}{2^2} + \sum_{n \text{ odd}} \frac{1}{n^2} < \frac{1}{2}.$$

Hence for some i, the row $a+i$ contains at least $n/2$ primes $p \geq \varepsilon n^2$. For $n > \varepsilon^{-1}$, none of the primes divide two numbers of the form $b+j$, so these $n/2$ primes are all distinct. All these primes divide $a+i$, so

$$a + i > (\varepsilon n^2)^{n/2} = \varepsilon^{n/2} \cdot n^n,$$

as needed.

> **Remark.** We prove $\min\{a, b\} > (cn)^n$ instead. The requested bound $(cn)^{n/2}$ is derived by proving primes $p < n$ cover at most $n^2/2$ cells, and using that estimate instead.

§6 Solutions to practice problems

§6.1 Solution 4.1

See Solution 3.2 above.

§6.2 Solution 4.2 (Iran 2001/3/2)

The answer is no. Assume FSoC such a sequence exists, and let P_n denote the first n primes. By the Principle of Inclusion-Exclusion,

$$\frac{1}{n} - a_n = \lim_{k\to\infty} \sum_{T \subset P_k} \frac{(-1)^{|T|}}{n \prod_{p \in T} p} = \frac{1}{n} \lim_{k\to\infty} \sum_{T \subset P_k} \frac{(-1)^{|T|}}{\prod_{p \in T} p} = \frac{1-a_1}{n}.$$

Solving, $a_n = \frac{1}{n} a_1$ for all n, so

$$1 = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{a_1}{k} = a_1 \sum_{k=1}^{\infty} \frac{1}{k}.$$

Since $\sum_{k=1}^{\infty} a_k$ diverges, $a_1 = 0$, which means that $a_n = 0$ for all n. This yields the desired contradiction.

§6.3 Solution 4.3 (Canada 2020/4)

TODO

§6.4 Solution 4.4 (ISL 2015 N6)

It turns out that f can be decomposed into finitely many infinite arithmetic sequences.

We begin by exercising condition (i).

> **Claim 1.** f is increasing and injective.

Proof. By (i), $f(m) - m \geq 1$. If $f(a) = f(b)$, then $a \equiv f^n(a) = f^n(b) \equiv b \pmod{n}$ for all n, so $a = b$. □

Thus, we can decompose f into chains of the form $m \to f(m) \to f^2(m) \to \cdots$. We will now interpret condition (ii).

> **Claim 2.** There are finitely many chains.

Proof. Note that $\mathbb{N} \setminus \{f(n) : n \in \mathbb{N}\}$ contains the heads of all the chains, so there are only finitely many of them. □

The key claim:

> **Claim 3 (USAMO 1995/4).** Each chain is an arithmetic sequence or has density zero.

Proof. We know each chain is infinite and increasing. Let q_0, q_1, ... be a chain, so by (i), $m-n$ divides $q_m - q_n$ for all m, n.

Let Q be the linear polynomial with $Q(0) = q_0$, $Q(1) = q_1$. Scale everything up so that Q has integer coefficients. Then $Q(n) \equiv Q(i) = q_i \equiv q_n \pmod{n-i}$ for $i = 0, 1$, so by Chinese Remainder theorem

$$Q(n) \equiv q_n \pmod{n(n-1)}.$$

Now,

- If $Q(n) = q_n$ for infinitely many n, then for all m and n with $Q(n) = q_n$, we have $Q(m) \equiv Q(n) = q_n \equiv q_m \pmod{n-m}$. By taking n large, we have $Q(m) = q_m$ always, i.e. the chain is an arithmetic sequence.
- Otherwise $q_n = Q(n) + kn(n-1) \geq O(n^2)$ for sufficiently large n, so the chain has density zero.

□

Now by Chinese Remainder theorem, the chains that are arithmetic sequences cover all integers larger than N for some N, so no chains of density 0 can exist. Therefore f is comprised of arithmetic sequences, and the problem condition readily follows.

§6.5 Solution 4.5 (China TST 2018/1/5)

Say a number is δ-good if at least $(1-\delta)n+1$ of the binomials are divisible by k. We first solve the problem for prime powers k:

> **Claim.** For each $\varepsilon, \delta > 0$, the density of δ-good numbers is at least $1 - \varepsilon$.

Proof. Let $n = \overline{a_j a_{j-1} \cdots a_0}_p$, and let S_t be the tth symmetric sum of $a_\bullet + 1$.

First we upper bound the number of i with $\nu_p\left(\binom{n}{i}\right) \leq e$. By Kummer's theorem, we can overcount the number of i by counting the number of i such that when written in base p, at most e digits are greater than the corresponding digit in n. This overcount describes at most $p^e S_{j+1-e}$ such i.

But if μ denotes the number of zeros in the base-p representation of n, then from $a_\bullet + 1 \leq p$,

$$S_{j+1-e} \leq \binom{j+1}{j+1-e} \cdot p^{j+1-e-\mu} = \binom{j+1}{e} \cdot p^{j+1-\mu}.$$

To consider the density of n, we wonder how often $p^e S_{j+1-e} \geq \delta n$. Substituting the upper bound for s_{j+1-e}, we find it is necessary for

$$p^\mu \leq 100 p^{e+2} \binom{j+1}{e} < 100 p^{e+2} (j+1)^e.$$

Hence $\mu < \log_p 100 + e + 2 + e \log_p (j+1) = O(\log\log n)$.

Then let $c = O(\log\log N) = O(\log m)$ be the upper bound for μ over all $n = 1, \ldots, N = p^m$; then each δ-bad n contains at most c zeros, so the number of δ-bad n is at most

$$\binom{m}{c} p^c (p-1)^{m-c} < m^c p^c (p-1)^{m-c} < \varepsilon p^m$$

for large enough m.

Digression: We can verify that $m^c p^c (p-1)^{m-c}$ is tiny compared to p^m as follows: rewrite the inequality as

$$\left(m \cdot \frac{p-1}{p}\right)^c \overset{?}{\ll} \left(\frac{p}{p-1}\right)^m,$$

then observe that asymptotically $x^{\log x} \ll c^x$.

Anyway, this shows at most ε of $n \le N$ are δ-bad, as claimed. □

Finally, let $k = p_1^{e_1} \cdots p_\ell^{e_\ell}$. We will show the density of k-good numbers is also 1. By the claim, for all $\varepsilon, \delta > 0$, the δ-good numbers with $k \to p_i^{e_i}$ have density at least $1 - \varepsilon$.

It follows tha the density of $\ell\delta$-good numbers for $k \to p_1^{e_1} \cdots p_\ell^{e_\ell}$ is at least $1 - \ell\varepsilon$, so the conclusion follows.

§6.6 Solution 4.6 (InfinityDots 2019/5)

Yes — in fact we describe S a subset of the unit circle. By perspectivity from $(0,1)$ to the x-axis, it suffices to find a set T of points on the number line that forms at least $|T|^2$ harmonic bundles.

I claim $T = \{0, 1, 2, \dots, n\}$ works for sufficiently large n. It is not hard to check that if $\nu_p(r) = 1$, where p is prime, then

$$\left(x, x + 2ri; x + \left(r + p^2\right) i, x + \left(r + \frac{r^2}{p^2}\right) i\right) = -1.$$

Each of these harmonic bundles is uniquely characterized by (x, r, i), since $\gcd\left(r + p^2, r + r^2/p^2\right) = 1$.

For each r, as x, i vary we have exhibited

$$\sum_{k \ge 1} \max\left(0, n - k\left(r + \frac{r^2}{p^2}\right)\right) \ge \frac{n}{2}\left(\frac{n}{r + \frac{r^2}{p^2}} - 1\right) = \frac{n^2}{2\left(r + \frac{r^2}{p^2}\right)} - \frac{n}{2}$$

harmonic bundles. Let

$$X(p) = \frac{n^2}{2} \sum_{\nu_p(r)=1} \max\left(0, \frac{1}{r + \frac{r^2}{p^2}} - \frac{1}{n}\right).$$

We have exhibited at least $X(p)$ harmonic bundles for each p.

As $r = ps$ varies, with $s > p$ and $p \nmid s$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{X(p)}{n^2} &= \lim_{n\to\infty} \frac{1}{2} \sum_{p \nmid s} \max\left(0, \frac{1}{ps + s^2} - \frac{1}{n}\right) \\
&\ge \lim_{n\to\infty} \left[\frac{1}{2p} \sum_{\substack{p \nmid s \\ s^2 \le n}} \left(\frac{1}{s} - \frac{1}{s+p}\right) - \frac{\sqrt{n}}{n}\right] \\
&= \frac{1}{2p}\left(\frac{1}{p+1} + \cdots + \frac{1}{2p}\right) \\
&\ge \frac{1}{2p} \cdot \frac{p}{2p} = \frac{1}{4p}.
\end{aligned}$$

There are at least $X(2)+X(3)+X(5)+\cdots$ harmonic bundles in T, but

$$\lim_{n\to\infty} \frac{X(2)+X(3)+X(5)+\cdots}{n^2} \geq \frac{1}{4}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots\right),$$

which diverges. (Note that the $n^{-1/2}$ term in the computation of $\lim_{n\to\infty} X(p)/n^2$ may be ignored since there are $O(\ln n)$ of them.) Hence for sufficiently large n, we have $X(2)+X(3)+X(5)+\cdots > n^2$, and we are done.

> **Remark.** In fact the above solution proves for each $k>0$, there is an S which forms at least $k|S|^2$ cyclic harmonic quadrilaterals.

§6.7 Solution 4.7 (ZhiHu)

> **Remark.** Source: example 3 on https://zhuanlan.zhihu.com/p/26101898.

Select very large primes $q_1, q_2, \ldots, q_{2020}$ with minimum q, and let $d(k)$ be the number of divisors of k. We will ensure that $d(n+i)$ is divisible by q_i for each i via CRT, then ensure no other $d(n+j)$ is divisible by q_i via density bounding.

Step I: CRT. Pick some more primes $p_1, \ldots, p_{2020}$ (each greater than 2020), and force $n \equiv p_i^{q_i-1} - i \pmod{p_i^{q_i}}$ for each i. By Chinese Remainder theorem, there are A and B such that all $n \equiv A \pmod{B}$ have this property.

Step II: Density bounding. By ensuring $p_i > 2020$, we have $p_i^{q_i-1} \nmid n+j$ for all other $j \neq i$. It suffices to ensure $p^{q-1} \nmid n+j$ for all choices of j and prime $p \notin \{p_1, \ldots, p_{2020}\}$. We will show this does not happen with density less than 1.

Fix p. The number of $n \leq N$ with $n \equiv A \pmod{B}$ for which there is some j with $n \equiv -j \pmod{p^{q-1}}$ is at most

$$\frac{2020}{Bp^{q-1}}N + 1.$$

Hence, the number of $n \leq N$ with $n \equiv A \pmod{B}$ for which some p exists with the above property is at most

$$\sum_p \left(\frac{2020}{Bp^{q-1}}N+1\right) \leq \frac{2020N}{B}\sum_p \frac{1}{p^{q-1}} + \pi(N) \ll \frac{N}{B}$$

by choosing p, q large enough.