

Advanced Counting Arguments

Based on lectures by CMU Professor John Mackey

Eric Shen

Friday, May 8, 2020

Contents

1	Understanding generating functions	2
2	Modeling sequences with generating functions	2
2.1	Definitions	2
2.2	Dealing with sums	3
2.3	Linear recurrence	3
2.4	Proving identities	4
2.5	Switching the order of summation	4
3	Partitions	5
3.1	Definitions	5
3.2	The Euler function	5
4	D.I.E.	6
4.1	The main idea	6
4.2	More walkthroughs	6
5	Generating functions, again	8
6	Problems	10
6.1	Generating functions problems	10
6.2	Summation problems	11
6.3	Partitions problems	11
6.4	D.I.E. problems	11
7	Solutions to walkthroughs	12
7.1	Solution 1.1 (JMO 2019/1)	12
7.2	Solution 2.2	12
7.3	Solution 2.3	12
7.4	Solution 2.4	13
7.5	Solution 3.1	13
7.6	Solution 3.2	13
7.7	Solution 4.2	14
7.8	Solution 4.3	14
7.9	Solution 4.4	14
7.10	Solution 5.1 (USAMO 2017/2)	15

§1 Understanding generating functions

These are merely glorified power series.

We use the following example to get a grasp of generating functions. Here, the generating function is just a way to imagine a multiset.

Example 1.1 (JMO 2019/1)

There are $a + b$ bowls arranged in a row, numbered 1 through $a + b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear.

A legal move consists of moving an apple from bowl i to bowl $i + 1$ and a pear from bowl j to bowl $j - 1$, provided that the difference $i - j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

Walkthrough. A typical generating function construction might be this: consider

$$A(X) = \sum_{\text{apples}} X^{\text{bowl}-1} \quad \text{and} \quad B(X) = \sum_{\text{pears}} X^{\text{bowl}-1}.$$

- (a) What are the initial values of $A(X)$, $B(X)$? What should they be at the end of the process so that we accomplish our goal?
- (b) Explain each “move” in the language of generating functions.

Let’s construct ab even.

- (c) Check that we may assume a is even without losing generality. We will attempt induction on b .
- (d) Prove that we can perform a series of moves so that $A(X)$ maps to $X \cdot A(X)$ and $B(X)$ maps to $B(X) + X - X^{a+1}$.
- (e) Apply the inductive hypothesis while ignoring the X term in $B(X)$.

Now let’s see why generating functions slaughter the proof of necessity. Assume a , b are both odd.

- (f) We will want an invariant expression in terms of A , B for which initial values and ending values are not equal. Why might an expression of the form $A(\lambda) + B(\lambda)$ not work?
- (g) Also see why $A(1) - B(1)$ does not work.
- (h) Modify the argument of (g) to find a λ so that $A(\lambda) - B(\lambda)$ is invariant, but the starting and ending values are not. Conclude.

§2 Modeling sequences with generating functions

§2.1 Definitions

Given a sequence a_0, a_1, a_2, \dots , we define an associated **generating function**

$$A(x) = \sum_{n \geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

For instance, the generating function for $b_0 = b_1 = b_2 = \dots = 1$ has **closed form**

$$B(x) = \sum_{n \geq 0} x^n = \frac{1}{1-x}.$$

§2.2 Dealing with sums

Example 2.1

Verify the following generating function identities:

- (a) $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$
- (b) $(1-x)^{-n} = \sum_{k \geq 0} \binom{n+k-1}{k} x^k.$
- (c) $\frac{1}{1-y(1+x)} = \sum_{k,n} \binom{n}{k} x^k y^n.$

Proof. Part (a) is just the binomial theorem.

For part (b), check that

$$\frac{1}{(1-x)^n} = \left(\sum_{k \geq 0} x^k \right)^n = \sum_{k \geq 0} \binom{n+k-1}{k} x^k$$

by binomial theorem.

For part (c), check that

$$\frac{1}{1-y(1+x)} = \sum_{n \geq 0} y^n (1+x)^n = \sum_{n \geq 0} y^n \sum_{k=0}^n \binom{n}{k} x^k = \sum_{k \leq n} \binom{n}{k} x^k y^n.$$

□

What we exhibited in the proof of (c) is a tactic known as *switching the order of summation*.

§2.3 Linear recurrence

Example 2.2

Determine a closed form for the sequence $a_0 = 1$, $a_1 = 6$, $a_{n+2} = 4a_{n+1} - 4a_n$.

Walkthrough. We consider the generating function

$$F(x) = \sum_{n \geq 0} a_n x^n.$$

- (a) In the definition of $F(x)$, express a_2, a_3, \dots in terms of previous terms of the sequence.
- (b) Express $F(x)$ in terms of $F(x)$, and solve for $F(x)$.
- (c) Expand the closed form using Example 2.1, and determine the coefficient of x^n in $F(x)$ for each n .

§2.4 Proving identities

Example 2.3

Prove the following three identities.

$$(a) \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

$$(b) \sum_{k=0}^n k \binom{n}{k} = n \cdot 2^{n-1}.$$

$$(c) \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

Walkthrough. Here are hints:

- (a) For part (a), consider $(1+x)^{2n} = (1+x)^n \cdot (1+x)^n$, and consider the x^n term.
- (b) For part (b), differentiate both the closed form and the expansion of the generating function $(1+x)^n$.
- (c) For part (c), consider $F(x) = (1+x)^n$, and check $F(1)$, $F(-1)$.

§2.5 Switching the order of summation

Example 2.4

For $n \geq 0$, compute

$$\sum_{k \geq 0} \binom{n+k}{2k} 2^{n-k}.$$

Walkthrough. The main idea is to consider the generating function

$$F(x) = \sum_{n \geq 0} \left[\sum_{k \geq 0} \binom{n+k}{2k} 2^{n-k} \right] x^n.$$

- (a) Switch the order of summation, pulling the $\sum_{k \geq 0}$ to the front.
- (b) Check that

$$F(x) = \sum_{k \geq 0} x^k \sum_{n \geq 0} \binom{(n-k)+2k}{2k} (2x)^{n-k}.$$
- (c) Substitute m for $n-k$ and remove the $\sum_{n \geq 0}$, so that k is the only remaining dummy variable in the summation.
- (d) Use the geometric series formula to get $F(x)$ as a rational function.
- (e) Apply partial fraction decomposition to split $F(x)$ into linear terms, and evaluate each term.
- (f) Determine the answer.

§3 Partitions

§3.1 Definitions

A **partition** of n is a way of writing n as the sum of the elements of a multiset of positive integers. As such, the number of partitions $\mathbf{P}(x)$ may be expressed in the form

$$\begin{aligned}\mathbf{P}(x) &= (1 + x + x^2 + \cdots) (1 + x^2 + x^4 + \cdots) (1 + x^3 + x^6 + \cdots) \\ &= \prod_{n \geq 1} \frac{1}{1 - x^n}.\end{aligned}$$

Example 3.1

Let n be a positive integer. Show that the number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

Walkthrough. As expected, we prove their generating functions are the same.

- Prove that the generating function for the number of partitions of n into distinct parts is given by $\prod_{n \geq 1} (1 + x^n)$.
- Prove that the generating function for the number of partitions of n into odd parts is given by $\prod_{n \geq 1} (1 - x^{2n-1})^{-1}$.
- Divide both of these by $\mathbf{P}(x)$ and show the quotients are equal.

§3.2 The Euler function

Example 3.2 (Pentagonal number theorem)

Prove that

$$\prod_{n \geq 1} (1 - x^n) = 1 + \sum_{k \geq 1} (-1)^k \left[x^{k(3k-1)/2} + x^{k(3k+1)/2} \right].$$

Walkthrough. (a) Where have we seen the left-hand expression before?

- Let $p(n)$ be the number of partitions of n and let the left-hand side be $\sum_{n \geq 0} a_n x^n$. Show that $\sum_{i=0}^n p(n-i)a_i = 0$ for all $n \geq 1$.
- We want to prove

$$a_i = \begin{cases} 1 & \text{if } i = \frac{1}{2}(3k^2 \pm k) \text{ and } k \text{ is even;} \\ -1 & \text{if } i = \frac{1}{2}(3k^2 \pm k) \text{ and } k \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

Determine a recurrence for $p(n)$ that is necessary and sufficient to show the above.

- Let $g_i := \frac{1}{2}(3i^2 - i)$, and let $\mathcal{P}(n)$ be the set of partitions of n . Let

$$\mathcal{X} := \bigcup_{i \text{ even}} \mathcal{P}(n - g_i) \quad \text{and} \quad \mathcal{Y} := \bigcup_{i \text{ odd}} \mathcal{P}(n - g_i).$$

Use (c) to check it suffices to show $|\mathcal{X}| = |\mathcal{Y}|$.

- (e) It suffices to find a bijective function $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$. Check that the following function which maps the partition $\mathcal{P}(n - g_i) \ni \lambda : n - g_i = \lambda_1 + \dots + \lambda_k$ to the partition $\lambda' = \varphi(\lambda)$ works:

$$\varphi(\lambda) := \begin{cases} \lambda' : n - g_{i+1} = (\lambda_2 + 1) + \dots + (\lambda_k + 1) + \underbrace{1 + \dots + 1}_{\lambda_1 - k - 3i} & \text{if } k + 3i \leq \lambda_1; \\ \lambda' : n - g_{i-1} = (k + 3i - 2) + (\lambda_1 - 1) + \dots + (\lambda_k - 1) & \text{if } k + 3i > \lambda_1. \end{cases}$$

§4 D.I.E.

§4.1 The main idea

D.I.E. stands for **Description, Involution, Exceptions**. A basic rundown of this method is as follows: suppose we are evaluating the sum

$$A_0 - A_1 + A_2 - A_3 + \dots .$$

We *describe* a set S_i with cardinality A_i for each i , then we provide an *involution* where we match positive-signed objects (e.g. A_i with i even in our example) with negative-signed objects (A_i with i odd). Finally, we add up the *exceptions* — the unmatched objects, using the appropriate sign.

The core idea is the *involution*, which we usually think of as “togglng” something. Here is an example:

Example 4.1

Prove that $\sum_{i=0}^N (-1)^i \binom{N}{i} = 0$.

Proof. Let $[N] = \{1, \dots, N\}$, and let $S_i = \{S \subseteq [N] : |S| = i\}$. We define an involution F from the set of subsets of $[N]$ to itself by $F(S) \equiv S \oplus \{1\}$; that is, $F(S) = S \sqcup \{1\}$ if $1 \notin S$ and $F(S) = S \setminus \{1\}$ if $1 \in S$.

Matched pairs in the involution correspond to terms that sum to 0. There are no exceptions, so the sum evaluates to 0. \square

Example 4.2

Evaluate $\sum_{i=0}^K (-1)^i \binom{N}{i}$.

Walkthrough. Consider the same involution as in Example 4.1, except some of the subsets of $[N]$ are missing.

- (a) Determine which subsets of $[N]$ remain unmatched.
- (b) Sum up the exceptions.

§4.2 More walkthroughs

Example 4.3

Let $N \leq K \leq M + N$. Evaluate

$$\sum_{i=0}^n (-1)^i \binom{N}{i} \binom{M + N - i}{K - i}.$$

Walkthrough. We let $S_i = \{(X, Y) : X \subseteq [N], |X| = i, Y \subseteq [M + N] \setminus X, |Y| = K - i\}$.

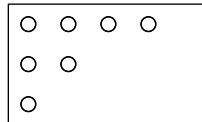
- (a) Figure out what the definition of S_i means.
- (b) Check that X, Y are disjoint, and determine an involution such that $X \cup Y$ remains invariant, but the size of X changes modulo 2.
- (c) Determine the exception.

Example 4.4

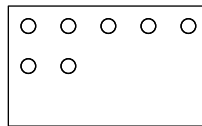
Let A be the number of partitions of N into an even number of parts, all distinct, and let B be the number of partitions of N into an odd number of parts, all distinct. Show that the coefficient of x^N in $(1 - x)(1 - x^2)(1 - x^3) \cdots$ is $A - B$.

Walkthrough. This is a more clever application of D.I.E.

- (a) Recall that the x^N coefficient of $(1 - x)(1 - x^2) \cdots$ is.
- (b) Per usual, depict partitions in a **Ferrers diagram** (see below):

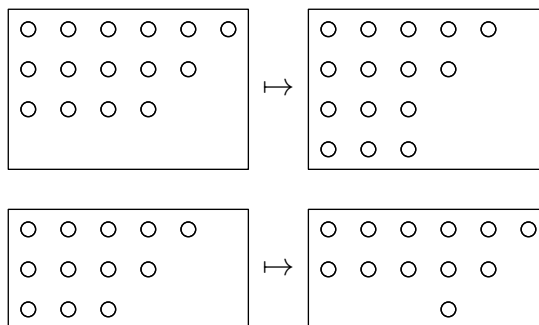


- (c) We want to construct an involution which toggles the parity of the number of rows. Determine involutions which map the above partition to this:



Let the *base* be the bottom row, and define the *slope* as follows: take the upper-right circle, then keep on moving southwest as far as possible. For instance, in both the above pictures, the slope has size 1.

- (d) Consider the following involution: let s, b be the sizes of the slope, base. If $s < b$, then move the slope under the base, and if $s \geq b$, move the base up along the slope. Check that this results in a valid partition *most of the time*.
- (e) Describe all exceptions. Shown below are some specific ones:



- (f) Exceptions are either
 - $s = b$ and the slope, base share a common point; or

- $s = b - 1$ and the slope, base share a common point.

These exceptions are only possible for certain n . Find all such n , and determine $A - B$ for each. Conclude the proof.

§5 Generating functions, again

Let's solve another problem through generating functions.

Example 5.1 (USAMO 2017/2)

Let m_1, m_2, \dots, m_n be n (not necessarily distinct) positive integers. For any sequence of integers $A = (a_1, \dots, a_n)$ and any permutation $w = (w_1, \dots, w_n)$ of (m_1, \dots, m_n) , define an A -inversion of w to be a pair of indices i, j with $i < j$ for which one of the following conditions holds:

- $a_i \geq w_i > w_j$,
- $w_j > a_i \geq w_i$, or
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, and for any positive integer k , the number of permutations of (m_1, \dots, m_n) having exactly k A -inversions is equal to the number of permutations of (m_1, \dots, m_n) having exactly k B -inversions.

Walkthrough. We first assume all $m_1 < \dots < m_n$ are distinct. In what follows, an *inversion* is a pair of indices i, j with $i < j$ and $w_i > w_j$.

- (a) By choosing suitable B , show that the task is to show the number of A -inversions is always equal to the number of regular inversions.

No explicit formula for the number of permutations with some number of inversions is known, but a generating function is known.

We will explicitly find it: let $n!_x$ be the generating function where the x^k coefficient is the number of permutations of $[n]$ having k inversions.

- (b) We find a recurrence: $n!_x$ in terms of $(n-1)!_x$. Now select the first element w_1 of the permutation. How many additional inversions are caused by w_1 ?
- (c) Use (b) to check that $n!_x = (1 + x + \dots + x^{n-1})(n-1)!_x$. Hence,

$$n!_x = 1 \cdot (1 + x) \cdot (1 + x + x^2) \cdots (1 + x + \dots + x^{n-1}).$$

Now we will show, using the same idea, that the generating function for the number of A -inversions is also $n!_x$.

- (d) Proceed by induction, with $n = 1$ clear. Assume the number of B -inversions of $[n-1]$ is $(n-1)!_x$ for all B , and again, fix w_1 . Also assume that $m_k \leq a_1 < m_{k+1}$ (with $m_0 = -\infty$, $m_{n+1} = \infty$).
- (e) Show that if $w_i = m_{k-i}$, $i \geq 0$, then there are $n - i - 1$ inversions with w_1 , and show that if $w_i = m_{k+i}$, $i > 0$, then there are $i - 1$ inversions with w_1 .

(f) Conclude that the generating function for the number of A -inversions of $[n]$ is $n!_x$.

Now we tackle the case where m_1, \dots, m_n are not necessarily distinct. Let the multiset $\{m_1 \leq m_2 \leq \dots \leq m_n\}$ contains k distinct integers $\lambda_1 < \dots < \lambda_k$ with multiplicity c_1, \dots, c_k .

Let $F(x)$ be the generating function for the number of permutations of (m_1, \dots, m_n) having exactly k A -inversions, with multiplicity (so $F(1) = n!$).

(g) Slightly perturb each m_i , so that they are all distinct. From our work above, determine the generating function $F(x)$ now.

(h) Undo each perturbation: show that the excessive ordering of the c_i instances of λ_i contribute an overcount of $c_i!_x/c_i!$.

(i) Show that

$$F(x) = n!_x \cdot \frac{c_1!c_2!\cdots c_k!}{c_1!_xc_2!_x\cdots c_k!_x},$$

and conclude that $F(x)$ is independent of A .

§6 Problems

§6.1 Generating functions problems

Problem 6.1 (China 1996). Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

Problem 6.2. Compute the probability of a sum of 14 when 3 dice are rolled.

Problem 6.3. Find an explicit formula for

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \binom{n}{9} + \cdots$$

in terms of n .

Problem 6.4 (360 Problems for Mathematical Contests). Find a closed-form expression for the sum

$$S_n = \binom{n}{1} - 3\binom{n}{3} + 5\binom{n}{5} - 7\binom{n}{7} + \cdots.$$

Problem 6.5. Determine an explicit form for a_n in the sequence $a_0 = 2$, $a_1 = 8$, and $a_{n+2} = \sqrt{a_{n+1}a_n}$ for $n \geq 0$.

Problem 6.6 (D.J. Newman). Suppose that a finite number of arithmetic sequences $a_1 + b_1n$, $a_2 + b_2n$, \dots , $a_k + b_kn$ partition the positive integers into disjoint subsets.

(a) If $b_1 \geq b_2 \geq \cdots \geq b_k$, show that $b_1 = b_2$.

(b) Prove that $\sum_{i=1}^k \frac{a_i}{b_i} = \frac{k-1}{2}$.

Problem 6.7 (HMMT 2007 C9). Let S denote the set of triples (i, j, k) of positive integers satisfying $i + j + k = 17$. Compute

$$\sum_{(i,j,k) \in S} ijk.$$

Problem 6.8 (CJMO 2020/1). Let N be a positive integer, and let S be the set of all tuples with positive integer elements and a sum of N . For instance, $t_1 = (N)$, $t_2 = (1, 1, N - 2)$, $t_3 = (1, N - 1)$, and $t_4 = (N - 1, 1)$ are all distinct tuples in S . For all tuples t , let $p(t)$ denote the product of all the elements of t . For instance, $p(t_1) = N$, $p(t_2) = N - 2$, and $p(t_3) = p(t_4) = N - 1$.

Evaluate the expression (where we sum over all elements t of S)

$$\sum_{t \in S} p(t).$$

Problem 6.9 (IMO 1995/6). Let p be an odd prime. Find the number of subsets A of $\{1, 2, \dots, 2p\}$ such that

- A has exactly p elements, and
- the sum all of the elements of A is divisible by p .

Problem 6.10. Compute

$$\sum_{i=0}^n (-1)^i \binom{n}{i}^2.$$

Problem 6.11 (USA TST 2010/8). Let $m \geq n \geq 1$ be integers. Prove that

$$\sum_{\substack{a_1 + \dots + a_n = m \\ a_i \geq 1}} 1^{a_1} 2^{a_2} 3^{a_3} \dots n^{a_n} = \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^m,$$

where the left-hand sum ranges over all n -tuples (a_1, \dots, a_n) of positive integers obeying $a_1 + \dots + a_n = m$.

§6.2 Summation problems

Problem 6.12. For $m, n \geq 1$, compute

$$\sum_{k \geq 0} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}.$$

Problem 6.13 (HMMT 2020 A4). Let $\mathcal{U}(n, k)$ be the number of distinct prime factors of n that are at least k . For example, $\mathcal{U}(90, 3) = 2$. Evaluate

$$\sum_{n \geq 1} \sum_{k \geq 1} \frac{\mathcal{U}(n, k)}{3^{n+k-7}}$$

to the nearest integer.

§6.3 Partitions problems

Problem 6.14. Find a bijective proof of Example 3.1.

Problem 6.15. Show that the number of partitions of n into j parts is the same as the number of partitions of n with largest part j .

Problem 6.16. Let $P(n, m)$ be the number of partitions of n into parts having size no larger than m . Find a recurrence relation for $P(n, m)$.

Problem 6.17. Show that the number of partitions of n into distinct odd parts is the same as the number of self-conjugate partitions of n .

§6.4 D.I.E. problems

Problem 6.18. Compute $\sum_{k=0}^n \binom{n}{k} \binom{k}{m} (-1)^k$.

Below, $f_0 = f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. For the following four problems, note the tiling interpretation of the Fibonacci numbers: f_n is the number of ways to tile a $1 \times n$ strip with 1×1 and 1×2 rectangles.

Problem 6.19. Compute $\sum_{k=1}^n (-1)^k f_k$.

Problem 6.20. Compute $\sum_{k=1}^n (-1)^k k f_k$.

Problem 6.21. Compute $\sum_{k=0}^n \binom{n-k}{k}$.

Problem 6.22. Compute $\sum_{k=0}^n (-1)^k \binom{n-k}{k}$.

Problem 6.23. Show that the number of derangements of $[n]$ is $\sum_{k=0}^n (-1)^k n! / k!$.

Problem 6.24. Let r be a fixed integer between 1 and n . Compute $\sum_{k=1}^n k \binom{n-k}{r-1}$.

Problem 6.25 (Iran TST 2020/2/6). Let $p = 2n + 1$ be an odd prime. Find all n -tuples $(x_1, x_2, \dots, x_n) \in (\mathbb{Z}/p\mathbb{Z})^n$ such that

$$\sum_{i=1}^n x_i \equiv \sum_{i=1}^n x_i^2 \equiv \dots \equiv \sum_{i=1}^n x_i^n \pmod{p}.$$

§7 Solutions to walkthroughs

§7.1 Solution 1.1 (JMO 2019/1)

The problem consists of two parts:

Proof of sufficiency: Assume ab is even, and since the problem is symmetric under reflection, without loss of generality let a be even. I will show by induction on b that the goal is always possible.

Consider $b = 0$ as the base case — there is nothing to show. Take the leftmost pear (in bowl $a + 1$), and move it all the way to bowl 1; when the pear moves from bowl n to bowl $n - 1$, also move the apple in bowl $a + 2 - n$.

Each apple moves exactly once, and so after this process, the leftmost bowl contains a pear, the next a bowls each contain an apple, and the rightmost $b - 1$ bowls contain a pear. Then apply the induction hypothesis on $b - 1$ to sort the remaining $a + b - 1$ apples.

Proof of necessity: Assume ab is odd. We can check that in total, ab moves occur. There are two types of moves:

- Moving two fruit from odd-numbered bowls to even-numbered bowls.
- Moving two fruit from even-numbered bowls to odd-numbered bowls.

Since in both the beginning and the end of the process, each bowl contains a fruit, the two types of moves occur equally often. This is absurd, since ab is odd.

§7.2 Solution 2.2

Consider

$$\begin{aligned} F(x) &:= \sum_{n \geq 0} a_n x^n \\ &= 1 + 6x + 4 \sum_{n \geq 2} a_{n-1} x^n + 4 \sum_{n \geq 2} a_{n-2} x^n \\ &= 1 + 6x + 4x(F(x) - 1) - 4x^2 F(x) \end{aligned}$$

This solves to

$$\begin{aligned} F(x) &= \frac{1 + 2x}{(1 - 2x)^2} = (1 + 2x) \sum_{n \geq 1} n 2^{n-1} x^{n-1} \\ &= \sum_{n \geq 1} n 2^{n-1} x^{n-1} + \sum_{n \geq 0} n 2^n x^n \\ &= \sum_{n \geq 0} 2^n (2n + 1) x^n. \end{aligned}$$

Answer: $2^n(2n + 1)$.

§7.3 Solution 2.3

- For part (a), consider $(1 + x)^{2n} = (1 + x)^n \cdot (1 + x)^n$, and consider the x^n term.
- For part (b), differentiate both the closed form and the expansion of the generating function $(1 + x)^n$.
- For part (c), consider $F(x) = (1 + x)^n$, and check $F(1)$, $F(-1)$.

§7.4 Solution 2.4

Consider

$$F(x) = \sum_{n \geq 0} \left[\sum_{k \geq 0} \binom{n+k}{2k} 2^{n-k} \right] x^n.$$

We have

$$\begin{aligned} F(x) &= \sum_{k \geq 0} x^{-k} \sum_{n \geq 0} \binom{n+k}{2k} (2x)^{n-k} = \sum_{k \geq 0} x^k \sum_{n \geq 0} \binom{(n-k)+2k}{2k} (2x)^{n-k} \\ &= \sum_{k \geq 0} x^k \sum_{m \geq 0} \binom{m+2k}{2k} (2x)^m = \sum_{k \geq 0} x^k \left(\frac{1}{1-2x} \right)^{2k+1} \\ &= \frac{1}{1-2x} \sum_{k \geq 0} \left[\frac{x}{(1-2x)^2} \right]^k = \frac{1-2x}{(1-2x)^2 - x} = \frac{1-2x}{(1-x)(1-4x)} \\ &= \frac{1/3}{1-x} + \frac{2/3}{1-4x} = \sum_{n \geq 0} \left(\frac{1}{3} + \frac{2}{3} 4^n \right) x^n. \end{aligned}$$

Answer: $\frac{1}{3} (2^{2n+1} + 2)$.

§7.5 Solution 3.1

Let $g(x)$ be the generating function for the number of partitions into distinct parts, and let $h(x)$ be that for partitions into odd parts. Observe that

$$\frac{g(x)}{\mathbf{P}(x)} = (1-x)(1+x)(1-x^2)(1+x^2) \cdots = (1-x^2)(1-x^4) \cdots = \frac{h(x)}{\mathbf{P}(x)}.$$

§7.6 Solution 3.2

Recall that $\mathbf{P}(x) \prod_{n \geq 1} (1-x^n) = 1$. Hence if $\mathbf{P}(x) = \sum_{n \geq 0} p(n)x^n$ and $\prod_{n \geq 1} (1-x^n) = \sum_{n \geq 0} a_n x^n$, we have $\sum_{i=0}^n p(n-i)a_i = 0$ for all $n \geq 1$.

The task is to prove

$$a_i = \begin{cases} 1 & \text{if } i = \frac{1}{2}(3k^2 \pm k) \text{ and } k \text{ is even;} \\ -1 & \text{if } i = \frac{1}{2}(3k^2 \pm k) \text{ and } k \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

Let $g_i := \frac{1}{2}(3i^2 - i)$. Proving the above is equivalent to proving $\sum_i (-1)^i p(n - g_i) = 0$.

Denote by $\mathcal{P}(n)$ the set of partitions of n . Let

$$\mathcal{X} := \bigcup_{i \text{ even}} \mathcal{P}(n - g_i) \quad \text{and} \quad \mathcal{Y} := \bigcup_{i \text{ odd}} \mathcal{P}(n - g_i).$$

It is sufficient to show $|\mathcal{X}| = |\mathcal{Y}|$. Observe that there is an involution $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ defined by

$$\varphi(\lambda) := \begin{cases} \lambda' : n - g_{i+1} = (\lambda_2 + 1) + \cdots + (\lambda_k + 1) + \underbrace{1 + \cdots + 1}_{\lambda_1 - k - 3i} & \text{if } k + 3i \leq \lambda_1; \\ \lambda' : n - g_{i-1} = (k + 3i - 2) + (\lambda_1 - 1) + \cdots + (\lambda_k - 1) & \text{if } k + 3i > \lambda_1. \end{cases}$$

It is a bijection, so we are done.

§7.7 Solution 4.2

Let $S_i = \{S \subseteq [N] : |S| = i\}$. We define an involution F from the set of subsets of $[N]$ to itself by $F(S) \equiv S \oplus \{1\}$; that is, F toggles whether $1 \in S$.

The exceptions are when $1 \notin S$ and $|S| = K$, so the answer is $(-1)^K \binom{N-1}{K}$.

§7.8 Solution 4.3

We proceed by D.I.E.

Description: Consider the set

$$S_i = \{(X, Y) : X \subseteq [N], |X| = i, Y \subseteq [M + N] \setminus X, |Y| = K - i\}.$$

We seek the sum $\sum (-1)^i |S_i|$.

Involution: Let $F : S_i \rightarrow S_i$ be defined by $F(X, Y) = (X \oplus \star, Y \oplus \star)$, where we let $\star = \min([N] \cap (X \cup Y))$.

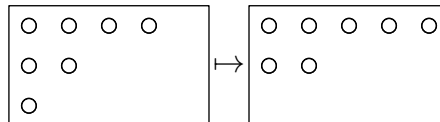
Exception: Evidently the only exceptions are those with $|X| = 0$ and $\min Y > n$; there are $\binom{M}{K}$ such exceptions, all with $i = 0$, so the answer is $\binom{M}{K}$.

§7.9 Solution 4.4

Recall the x^N coefficient of $(1 - x)(1 - x^2) \dots$ is

$$\begin{cases} 1 & \text{if } i = \frac{1}{2}(3k^2 \pm k) \text{ and } k \text{ is even;} \\ -1 & \text{if } i = \frac{1}{2}(3k^2 \pm k) \text{ and } k \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

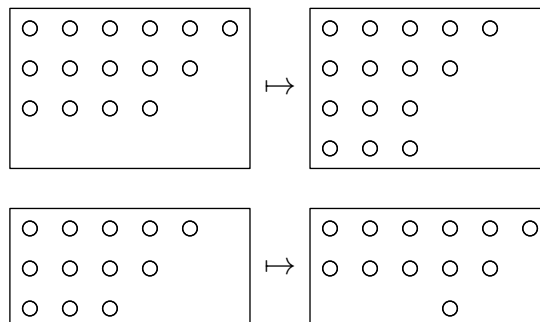
Use the standard row formation of partitions. Let the *base* be the bottom row, and define the *slope* as follows: take the upper-right circle, then keep on moving southwest as far as possible. For instance, in both the below pictures, the slope has size 1.



We use D.I.E., with the involution as follows: let s, b be the sizes of the slope, base. If $s < b$, then move the slope under the base, and if $s \geq b$, move the base up along the slope.

The exceptions are as follows:

- $s = b - 1$ and the slope, base share a common point; or
- $s = b$ and the slope, base share a common point.



The former only happens when $n = m^2 + \binom{m+1}{2}$, and the latter only happens when $n = m^2 - m + \binom{m+1}{2}$. The result follows.

§7.10 Solution 5.1 (USAMO 2017/2)

We will prove for any A that the number of permutations having exactly k A -inversions equals the number of permutations having exactly k inversions, where an inversion is a pair of indices i, j with $i < j$ and $a_i > a_j$.

The proof proceeds in two main steps:

- (i) Prove the problem for $m_1 < \dots < m_n$ (all distinct).
- (ii) Extend to all $m_1 \leq \dots \leq m_n$ (not necessarily distinct).

First, combinatorial proof of (i) Let $\text{Inv}_A(i)$ be the number of $j > i$ such that (w_i, w_j) is an A -inversion. We will construct a bijection between (w_1, \dots, w_n) with k A -inversions and (p_1, \dots, p_n) with k inversions. The process is as follows:

For each $i = n, \dots, 1$, write the variable p_i on the board such that p_i is the $(\text{Inv}_A(i) + 1)$ th element from the left. (For instance, we first write p_n on the board; then, if $\text{Inv}_A(n-1) = 0$, we write p_{n-1} to the left of p_n , and if $\text{Inv}_A(n-1) = 1$, we write p_{n-1} to the right of p_n .) Then set the i th variable from the left equal to m_i . (For instance, if the board reads p_3, p_1, p_2 , then we have $p_3 = m_1, p_1 = m_2, p_2 = m_3$.)

By design, for each i , the number of $j < i$ with $p_i < p_j$ equals $\text{Inv}_A(i)$. Hence the number of inversions of (p_1, \dots, p_n) equals the number of A -inversions of (w_1, \dots, w_n) .

This operation is clearly injective, thus it is a bijection, and the proof is complete.

Second, inductive proof of (i), with generating functions We claim via induction on n that the generating function for the number of permutations having k A -inversions is always

$$n!_x = 1 \cdot (1+x) \cdot (1+x+x^2) \cdots (1+x+\cdots+x^{n-1}).$$

(Here, the number of permutations having k A -inversions is the coefficient of x^k .)

The base case $n = 1$ is clear. Assume the claim is true for $n-1$, and let $m_k \leq a_1 < m_{k+1}$ (with $m_0 = -\infty, m_{n+1} = \infty$).

- For $i \geq 0$, if $w_1 = m_{k-i}$, then there are $n-i-1$ inversions with w_1 . (Namely (w_1, w_j) with $j < k-i$ or $j \geq k+1$.) Thus this case contributes a $x^{n-i-1}(n-1)!_x$ term.
- For $i > 0$, if $w_i = m_{k+i}$, then there are $i-1$ inversions with w_1 . (Namely (w_1, w_j) , with $k+1 \leq j < k+i$.) Thus this case contributes a $x^i(n-1)!_x$ term.

The new generating function is then

$$n!_x = (n-1)!_x \left(\sum_{i=0}^{k-1} x^{n-i-1} + \sum_{i=1}^{n-k} x^{i-1} \right) = (n-1)!_x (1+x+\cdots+x_{n-1}),$$

as needed.

Proof of (ii), with generating functions What follows is really a combinatorial argument, but expressing it without generating functions is a huge pain.

We reuse notation from the generating functions proof of (i), where $n!_x$ is the generating function for the number of permutations of (m_1, \dots, m_n) having k A -inversions. (It is not necessary to know the explicit form for $n!_q$ that we used above.)

Let the multiset $\{m_1, \dots, m_n\}$ contain k distinct integers $\lambda_1 < \dots < \lambda_k$ with multiplicity c_1, \dots, c_k respectively. Let $F(x)$ be the generating function for the number of permutations of (m_1, \dots, m_n) having k A -inversions, with multiplicity (so $F(1) = n!$).

Claim. The explicit form for $F(x)$ is

$$F(x) = n!_x \cdot \frac{c_1!c_2!\cdots c_k!}{c_1!_xc_2!_x\cdots c_k!_x}.$$

Proof. Slightly perturb each m_i , replacing m_i with $m_i + i\varepsilon$, where $0 < \varepsilon \ll 1/n$. Then the generating function is $n!_x$. (Obviously both proofs to (i) still hold when (m_1, \dots, m_n) are real numbers.)

Next we undo each perturbation. For $i = 1, \dots, k$, I claim the excessive ordering of the c_i instances of λ_i contribute an overcount of $c_i!_x/c_i!$. Indeed, there is a factor of $c_i!_x$ that should instead be $c_i! \cdot x^0$, since there are no inversions between equivalent elements of the permutation.

Undoing the overcounts, the claim then follows. \square

The x^k coefficient of $F(x)$ is the number of permutations having k A -inversions. This is independent of A , so we are done.