Geometry At Its Best

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§1 Problems

Problem 1 (IMO 2018/1)**.** Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of $\overline{BD}$ and $\overline{CE}$ intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or are the same line.

Problem 2 (APMO 2004/2)**.** Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, and COH is equal to the sum of the areas of the other two.

Problem 3 (Iran TST 2011/1)**.** In acute triangle ABC, let E and F be the feet of the altitudes from B and C, respectively. Let M, K, L denote the midpoints of $\overline{BC}$, $\overline{ME}$, $\overline{MF}$, respectively. Line KL intersects the line through A parallel to $\overline{BC}$ at T. Prove that $TA = TM$.

Problem 4 (USAMO 2010/1)**.** Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Problem 5 (ISL 2005 G1)**.** Let ABC be a triangle satisfying $AC+BC = 3AB$. The incircle of $\triangle ABC$ has center I and touches sides BC and CA at points D and E, respectively. Let K and L be the reflections of points D and E across I. Prove that points A, B, K, L are concyclic.

Problem 6 (Sharygin Correspondence 2012/8)**.** Let ABC be a right triangle with $\angle B = 90^\circ$, and let M be the midpoint of $\overline{AC}$. The incircle of triangle ABM touches sides AB and AM at points A_1 and A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of $\angle ABC$.

Problem 7 (ISL 2009 G4)**.** Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of $\overline{AB}$ and $\overline{CD}$ are G and H, respectively. Show that $\overline{EF}$ is tangent at E to the circle through the points E, G and H.

Problem 8 (Iran TST 2018/1/4)**.** Let ABC be a triangle with $\angle A \neq 90^\circ$, and let $\overline{BE}$ and $\overline{CF}$ be its altitudes. The bisector of $\angle A$ intersects $\overline{EF}$ and $\overline{BC}$ at M and N respectively. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that line AP bisects $\overline{BC}$.

Problem 9 (China Southeast 2018/5)**.** Let ABC be an isosceles triangle with $AB = AC$. Suppose that the center of circle ω is the midpoint of the $\overline{BC}$, and $\overline{AB}$ and $\overline{AC}$ are tangent to ω at points E and F respectively. There is a point G that lies on ω such that $\angle AGE = 90^\circ$. Show that if the tangent to ω at G meets $\overline{AC}$ at K, then line BK bisects $\overline{EF}$.

Problem 10 (EGMO 2020/3)**.** Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$. Prove that if the internal angle bisectors of $\angle A$, $\angle C$, $\angle E$ concur, then the internal angle bisectors of $\angle B$, $\angle D$, $\angle F$ concur.

Problem 11 (ISL 2004 G5)**.** Let $A_1A_2\cdots A_n$ be a regular n-gon. For $i = 1, \dots, n-1$, if $i = 1$ or $i = n-1$, then B_i is the midpoint of the side A_iA_{i+1}. Otherwise, if $S_i = \overline{A_1A_{i+1}} \cap \overline{A_nA_i}$, then B_i is intersection of the bisector of $\angle A_iS_iA_{i+1}$ with $\overline{A_iA_{i+1}}$. Prove that

$$\angle A_1B_1A_n + \angle A_1B_2A_n + \cdots + \angle A_1B_{n-1}A_n = 180^\circ.$$

Problem 12 (Iran TST 2010/5)**.** Circles ω_1 and ω_2 intersect at points P and K. Line XY is tangent to ω_1 at X and ω_2 at Y. Let lines YP and XP meet ω_1 and ω_2, respectively, again at B and C, respectively, and let lines BX and CY meet at A. Prove that if the circumcircles of $\triangle ABC$ and $\triangle AXY$ meet again at Q, then $\angle QXA = \angle QKP$.

Problem 13 (GOTEEM 2020/5). Let ABC be a triangle and let B_1 and C_1 be variable points on sides $\overline{BA}$ and $\overline{CA}$, respectively, such that $BB_1 = CC_1$. Let $B_2 \neq B_1$ denote the point on (ACB_1) such that $\overline{BC_1}$ is parallel to $\overline{B_1B_2}$, and let $C_2 \neq C_1$ denote the point on (ABC_1) such that $\overline{CB_1}$ is parallel to $\overline{C_1C_2}$. Prove that as B_1 and C_1 vary, the circumcircle of $\triangle AB_2C_2$ passes through a fixed point other than A.

Problem 14 (IMO 2008/6). Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the extension of ray BA past A, to the extension of ray BC past C, to the line AD, and to the line CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Problem 15 (CAMO 2020/3). Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Point M is the midpoint of $\overline{EF}$, and T is the point on ω such that $\overline{DT}$ is a diameter. Line MT meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines DF and DE intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.

Problem 16 (Serbia 2017/6). Let ABC be a triangle and let the common external tangents to the circumcircle and the A-excircle intersect line BC at P and Q. Show that $\angle PAB = \angle CAQ$.

Problem 17 (MOP 2019 + USA TST 2019/6). Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90°$. Denote by E and F the feet of the altitudes from B and C, respectively.

Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively. Prove that:

(a) (MOP 2019) If line EF is tangent to the incircle of $\triangle ABC$, then quadrilateral $AB_1A_1C_1$ is cyclic.

(b) (USA TST 2019/6) If quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

Problem 18 (ISL 2017 G7). A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_A, I_B, I_C, I_D be the incenters of the triangles DAB, ABC, BCD, CDA, respectively. Suppose that the common external tangents of the circumcircles of $\triangle AI_BI_D$ and $\triangle CI_BI_D$ meet at X, and the common external tangents of the circumcircles of $\triangle BI_AI_C$ and $\triangle DI_AI_C$ meet at Y. Prove that $\angle XIY = 90°$.

Problem 19 (IMO 2000/6). In acute triangle ABC, let H_1, H_2, H_3 be the feet of the altitudes from A, B, C, respectively, and let T_1, T_2, T_3 be the points where the incircle touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Prove that the reflections of $\overline{H_1H_2}$, $\overline{H_2H_3}$, $\overline{H_3H_1}$ over $\overline{T_1T_2}$, $\overline{T_2T_3}$, $\overline{T_3T_1}$, respectively, are the sides of a triangle that is inscribed in the incircle of $\triangle ABC$.

Problem 20 (Taiwan TST 2014/3/3). Let ABC be a triangle with circumcircle Γ and let M be an arbitrary point on Γ. Suppose that the tangents from M to the incircle of ABC intersect $\overline{BC}$ at two distinct points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 passes through the tangency point of the A-mixtilinear incircle with Γ.

Problem 21 (ISL 2018 G5). Let ABC be a triangle with circumcircle ω and incenter I. A line ℓ meets the lines AI, BI, and CI at points D, E, and F, respectively, all distinct from A, B, C and I. Prove that the circumcircle of the triangle determined by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ is tangent to ω.

Problem 22 (Iran TST 2017/3/6)**.** Let ABC be a triangle with circumcenter O and orthocenter H. Point P is the reflection of A with respect to $\overline{OH}$. Assume that P is not on the same side of $\overline{BC}$ as A. Points E and F lie on $\overline{AB}$ and $\overline{AC}$ respectively such that $BE = PC$ and $CF = PB$. Let K be the intersection of $\overline{AP}$ and $\overline{OH}$. Prove that $\angle EKF = 90^\circ$.

Problem 23 (USAMO 2016/3)**.** Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines I_BF and I_CE meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

Problem 24 (Sharygin 2014/10.8)**.** Let $ABCD$ be a cyclic quadrilateral. Prove that if the symmedian points of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, $\triangle DAB$ are concyclic, then quadrilateral $ABCD$ has two parallel sides.

Problem 25 (APMO 2019/3)**.** Let ABC be a scalene triangle with circumcircle Γ, and let M be the midpoint of $\overline{BC}$. A variable point P is selected on $\overline{AM}$. The circumcircles of triangles BPM and CPM intersect Γ again at points D and E, respectively. Lines DP and EP intersect the circumcircles of triangles CPM and BPM again at X and Y, respectively. Prove that as P varies, the circumcircle of $\triangle AXY$ passes through a fixed point T distinct from A.

Problem 26 (IMO 2018/6)**.** A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

Problem 27 (TSTST 2016/6)**.** Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

Problem 28 (IMO 2011/6)**.** Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ.

Problem 29 (Iran TST 2011/6)**.** Let ω be a circle with center O, and let T be a point outside of ω. Points B and C lie on ω such that $\overline{TB}$ and $\overline{TC}$ are tangent to ω. Select two points K and H on $\overline{TB}$ and $\overline{TC}$, respectively.

(a) Lines BO and CO meet ω again at B' and C', and points K' and H' lie on the angle bisectors of $\angle BCO$ and $\angle CBO$, respectively, such that $\overline{KK'}$ and $\overline{HH'}$ are perpendicular to $\overline{BC}$. Prove that K, H', B' are collinear if and only if H, K', C' are collinear.

(b) Let I be the incenter of $\triangle OBC$. Two circles in the interior of $\triangle TBC$ are externally tangent to ω and externally tangent to each other at J. Given that one of them is tangent to $\overline{TB}$ at K and the other is tangent to $\overline{TC}$ at H, prove that quadrilaterals $BKJI$ and $CHJI$ are cyclic.

Problem 30 (USA TST 2020/6)**.** Let $P_1P_2 \cdots P_{100}$ be a cyclic 100-gon, and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $P_{i-2}P_{i+1}$ and $P_{i-1}P_{i+2}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points Q_1, Q_2, …, Q_{100} are concyclic.

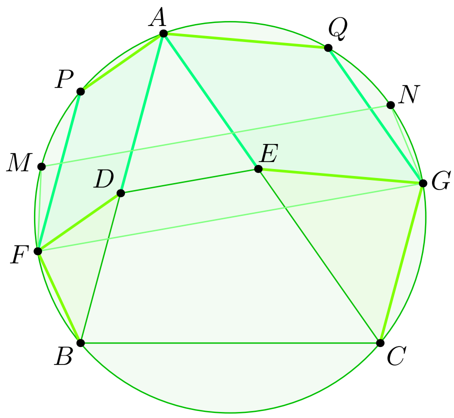
§2 Solutions

§2.1 IMO 2018/1 (Silouanos Brazitikos, Vangelis Psyxas, Michael Sarantis)

Problem 1 (IMO 2018/1)

Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of $\overline{BD}$ and $\overline{CE}$ intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or are the same line.

First solution, by constructing parallelograms Construct points P and Q on Γ such that $ABFP$ and $ACGQ$ are isosceles trapezoids, and let M and N be the midpoints of minor arcs AB and AC respectively. It is obvious that M and N are the midpoints of arcs PF and QG as well. Noting that $AP = BF = DF$ and $AQ = CG = EG$, we have $APFD$ and $AQGE$ are parallelograms.



By $PF = AD = AE = QG$, we have $\widehat{PF} = \widehat{QG}$ and thus $\widehat{MF} = \widehat{NG}$. It follows that $FGNM$ is an isosceles trapezoid, so $\overline{FG} \parallel \overline{MN}$. But both $\overline{DE}$ and $\overline{MN}$ are perpendicular to the internal angle bisector of $\angle A$, so $\overline{DE}$ and $\overline{FG}$ are parallel, as desired.

Second solution, by angle chasing Let $\overline{FD}$ and $\overline{GE}$ intersect Γ again at X and Y respectively. Notice that

$$\measuredangle AXD = \measuredangle AXF = \measuredangle ABF = \measuredangle DBF = \measuredangle FDB = \measuredangle XDA,$$

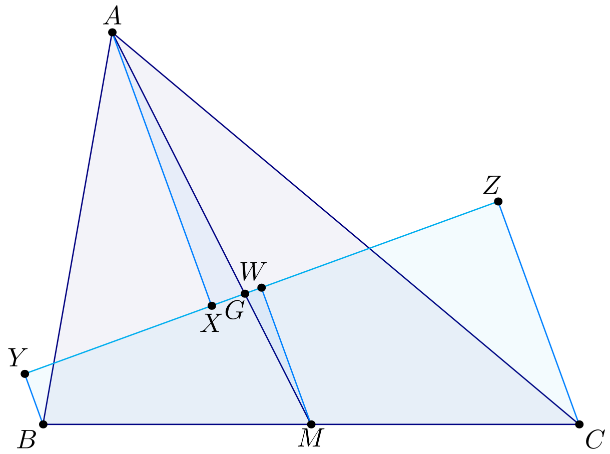
whence $AX = AD$. Analogously, $AY = AE$, so D, E, X, Y lie on a circle with center A. By Reim's theorem, $\overline{DE} \parallel \overline{FG}$, as desired.

§2.2 APMO 2004/2

Problem 2 (APMO 2004/2)

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, and COH is equal to the sum of the areas of the other two.

Assume WLOG that line OH does not intersect segment BC. Let G be the centroid of $\triangle ABC$, M the midpoint of $\overline{BC}$, and X, Y, Z, W the projections of A, B, C, M, respectively, onto line OH. Since G, O, H lie on the Euler Line, it suffices to prove that $AX = BY + CZ$.

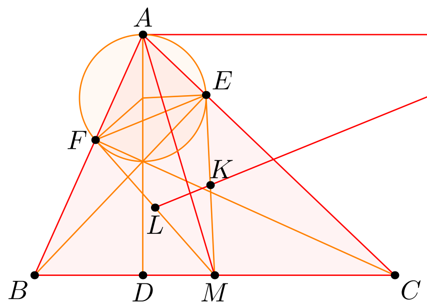


Since $\overline{MW} \parallel \overline{BY} \parallel \overline{CZ}$ and M is the midpoint of $\overline{BC}$, $\overline{MW}$ is the midline of trapezoid $BYZC$. However, by AA, $\triangle AGX \sim \triangle MGW$. Since $AG : GM = 2 : 1$, we have that $AX = 2 \cdot MW = BY + CZ$, as required.

§2.3 Iran TST 2011/1

Problem 3 (Iran TST 2011/1)

In acute triangle ABC, let E and F be the feet of the altitudes from B and C, respectively. Let M, K, L denote the midpoints of $\overline{BC}$, $\overline{ME}$, $\overline{MF}$, respectively. Line KL intersects the line through A parallel to $\overline{BC}$ at T. Prove that $TA = TM$.

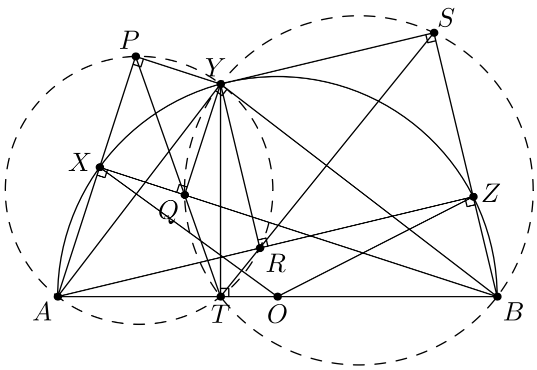


Let ω be the circle centered at M with radius 0. It is well-known that $\overline{TA}$, $\overline{ME}$, and $\overline{MF}$ are tangent to (AEF), so line KL is the radical axis of (AEF) and ω. It follows that $TA^2 = \text{Pow}(T, (AEF)) = \text{Pow}(T, \omega) = TM^2$, as desired.

§2.4 USAMO 2010/1 (Zuming Feng)

Problem 4 (USAMO 2010/1)

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.



Let T be the projection of Y onto $\overline{AB}$. Notice that T lies on the Simson Line $\overline{PQ}$ from Y to $\triangle AXB$, and the Simson Line $\overline{RS}$ from Y to $\triangle AZB$. Hence, $T = \overline{PQ} \cap \overline{RS}$, so it suffices to show that $\angle PTS = \frac{1}{2}\angle XOZ$.

Since $TAPY$ and $TBSY$ are cyclic quadrilaterals,

$$\angle PTS = \angle PTY + \angle YTS = \angle XAY + \angle YBZ = \frac{1}{2}\angle XOY + \frac{1}{2}\angle YOZ = \frac{1}{2}\angle XOZ,$$

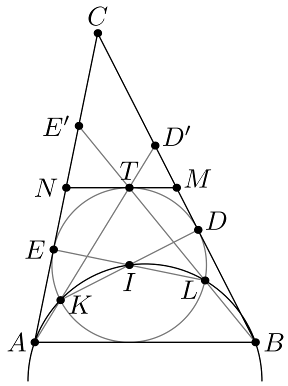
as required.

§2.5 ISL 2005 G1 (Dimitris Kontogiannis)

Problem 5 (ISL 2005 G1)

Let ABC be a triangle satisfying $AC + BC = 3AB$. The incircle of $\triangle ABC$ has center I and touches sides BC and CA at points D and E, respectively. Let K and L be the reflections of points D and E across I. Prove that points A, B, K, L are concyclic.

First solution, by Reim's theorem Let M and N be the midpoints of $\overline{CB}$ and $\overline{CA}$ respectively. By Pitot's theorem, $\overline{MN}$ is tangent to the incircle, say at a point T. Let the A-excircle touch $\overline{CB}$ at D' and the B-excircle touch $\overline{CA}$ at E'. By inspection, D, D', T lie on a circle centered at M, but by homothety A, K, D' are collinear. Thus $\angle ATD = \angle D'TD = 90^\circ = \angle KTD$, so A, K, T are collinear.



Similarly B, L, T are collinear, so A, B, K, L are concyclic by Reim's theorem. To spell it out, $\measuredangle AKL = \measuredangle TKL = \measuredangle MTL = \measuredangle ABL$.

Second solution, by Ptolemy's theorem Let $\overline{CI}$ intersect (ABC) again at T, and let I_C be the C-excenter. Note by the Incenter-Excenter Lemma that $TA = TI = TB$, and by Ptolemy's theorem on $ACBT$,

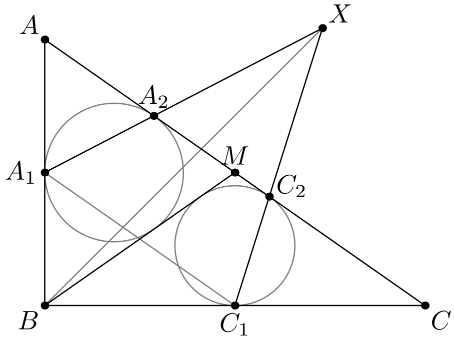
$$CT \cdot AB = TI(CA + CB) = 3TI \cdot AB \implies CI = 2IT = II_C,$$

whence I_C is the reflection of C across I. This implies that $\overline{KI_C}$ and $\overline{LI_C}$ are tangent to the incircle, so $\angle IKI_C = \angle ILI_C = 90^\circ$, and K and L lie on $(AIBI_C)$, as desired.

§2.6 Sharygin Correspondence 2012/8

Problem 6 (Sharygin Correspondence 2012/8)

Let ABC be a right triangle with $\angle B = 90°$, and let M be the midpoint of $\overline{AC}$. The incircle of triangle ABM touches sides AB and AM at points A_1 and A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of $\angle ABC$.



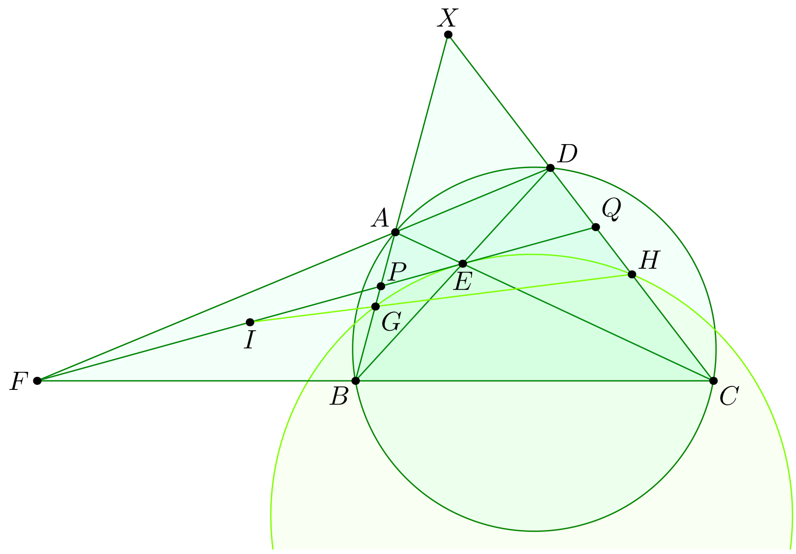
The key observation is that $\overline{A_1X}$ is the angle bisector of $\angle AA_1C_1$. Symmetrically applying this argument, X is the B-excenter of $\triangle A_1BC_1$, from which the result is obvious.

To show this, note that since M is the circumcenter of $\triangle ABC$, $\triangle MAB$ is isosceles, so A_1 is the midpoint of $\overline{AB}$. It follows that $2\measuredangle AA_1A_2 = \measuredangle A_1AA_2 = \measuredangle AA_1C_1$, as desired.

§2.7 ISL 2009 G4 (David Monk)

Problem 7 (ISL 2009 G4)

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of $\overline{AB}$ and $\overline{CD}$ are G and H, respectively. Show that $\overline{EF}$ is tangent at E to the circle through the points E, G and H.



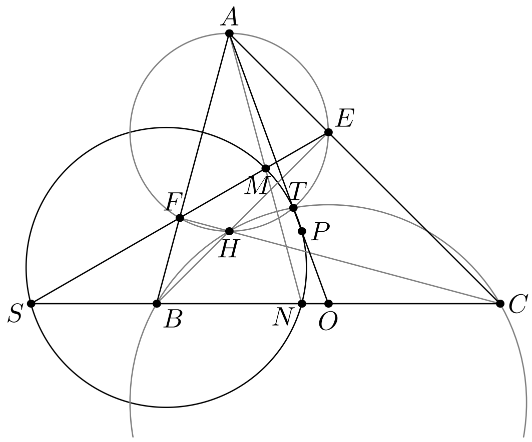
Let line EF intersect $\overline{AB}$ and $\overline{CD}$ at P and Q respectively, and let lines AB and CD meet at X. By Ceva-Menelaus, $-1 = (XP; AB) = (XQ; DC)$, so by the Midpoint of Harmonic Bundles Lemma, $XP \cdot XG = XA \cdot XB = XD \cdot XC = XQ \cdot XH$, thus $PGHQ$ is cyclic.

Denote by I the midpoint of $\overline{EF}$. Check that G, H, I lie on the Gauss line of $ACBD$. However it is well-known that $-1 = (EF; PQ)$, so by the Midpoint of Harmonic Bundles Lemma, $IE^2 = IP \cdot IQ = IG \cdot IH$. This completes the proof.

§2.8 Iran TST 2018/1/4 (Iman Maghsoudi)

Problem 8 (Iran TST 2018/1/4)

Let ABC be a triangle with $\angle A \neq 90^\circ$, and let $\overline{BE}$ and $\overline{CF}$ be its altitudes. The bisector of $\angle A$ intersects $\overline{EF}$ and $\overline{BC}$ at M and N respectively. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that line AP bisects $\overline{BC}$.



Let O be the midpoint of $\overline{BC}$, let H be the orthocenter, let T be the A-Humpty point, and denote $S = \overline{BC} \cap \overline{EF}$. By construction, $T \in \overline{AO}$ and $\overline{AO} \perp \overline{ST}$.

Since T lies on (HEF) and (HBC), T is the Miquel point of $BCFE$. Let Ψ be the spiral similarity at T sending $\overline{BC}$ to $\overline{EF}$. Note that

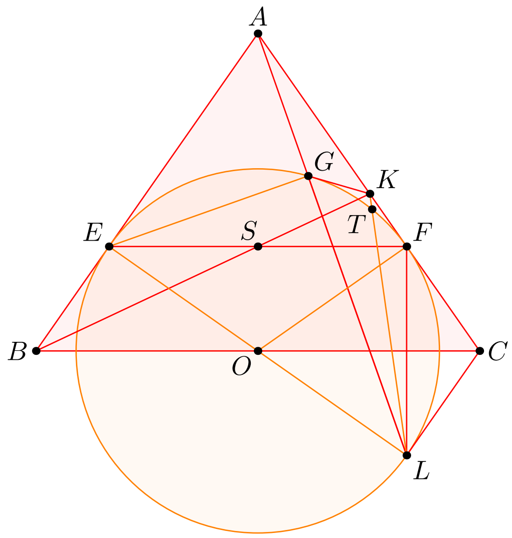
$$\frac{NB}{NC} = \frac{AB}{AC} = \frac{AE}{AF} = \frac{ME}{MF},$$

so Ψ sends N to M. It follows that $SMTN$ is cyclic, say with circumcircle Γ, but by definition, $\overline{SP}$ is a diameter of Γ. Thus $\angle STP = 90^\circ$, so P lies on $\overline{ATO}$, as desired.

§2.9 China Southeast 2018/5

Problem 9 (China Southeast 2018/5)

Let ABC be an isosceles triangle with $AB = AC$. Suppose that the center of circle ω is the midpoint of the $\overline{BC}$, and $\overline{AB}$ and $\overline{AC}$ are tangent to ω at points E and F respectively. There is a point G that lies on ω such that $\angle AGE = 90°$. Show that if the tangent to ω at G meets $\overline{AC}$ at K, then line BK bisects $\overline{EF}$.



Let line AG intersect ω again at L, $\overline{LK}$ intersect ω again at T, and $\overline{BK}$ intersect $\overline{EF}$ at S. Note that since $\angle EGL = 90°$, L is the antipode of E on ω, so F and L are reflections over $\overline{BC}$. It follows that $\overline{CL}$ is tangent to ω, whence

$$-1 = (GF;TL) \overset{L}{=} (AF;KC) \overset{B}{=} (EF;S\infty_{BC}).$$

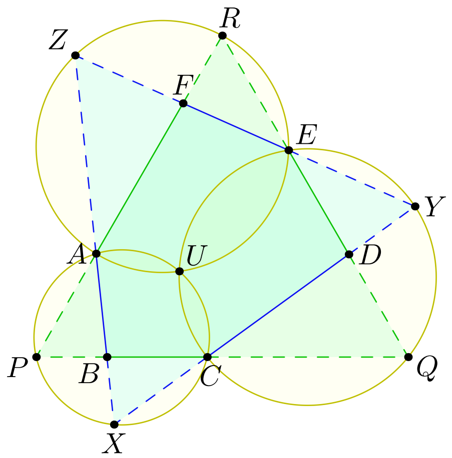
This implies that S is the midpoint of $\overline{EF}$, and we are done.

§2.10 EGMO 2020/3

Problem 10 (EGMO 2020/3)

Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$. Prove that if the internal angle bisectors of $\angle A$, $\angle C$, $\angle E$ concur, then the internal angle bisectors of $\angle B$, $\angle D$, $\angle F$ concur.

Evidently we can extend the sides of $ABCDEF$ to form two equilateral triangles PQR, XYZ as shown below.



It is easy to check that $APXC$, $CQYE$, $ERZA$ are cyclic, and by Miquel's theorem their circumcircles concur at a point U.

Claim 1. If the angle bisectors of $\angle A$, $\angle C$, $\angle E$ concur, then they concur at U.

Proof. Let them concur at U'. It is clear $\angle A + \angle B = 240°$, so $\angle AU'C = 360° - \angle B - \angle A = 120°$. It follows that U' lies on the circles $(APXC)$, etc., thus proving the claim. □

Claim 2. The angle bisectors of $\angle A$, $\angle C$, $\angle E$ concur at U if and only if $\triangle PQR \cong \triangle XYZ$.

Proof. Note that U is the spiral center sending $\triangle PQR$ to $\triangle XYZ$. However,

$$\measuredangle PAU = \measuredangle UAX \iff UP = UX \iff \triangle PQR \cong \triangle XYZ,$$

as needed. □

Symmetrially the angle bisectors of $\angle B$, $\angle D$, $\angle F$ concur if and only if $\triangle PQR \cong \triangle ZXY$. This completes the proof.

§2.11 ISL 2004 G5 (Dusan Dukic)

Problem 11 (ISL 2004 G5)

Let $A_1A_2\cdots A_n$ be a regular n-gon. For $i = 1, \dots, n-1$, if $i = 1$ or $i = n-1$, then B_i is the midpoint of the side A_iA_{i+1}. Otherwise, if $S_i = \overline{A_1A_{i+1}} \cap \overline{A_nA_i}$, then B_i is intersection of the bisector of $\angle A_iS_iA_{i+1}$ with $\overline{A_iA_{i+1}}$. Prove that

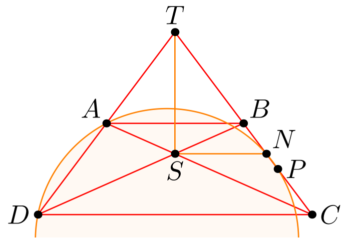
$$\angle A_1B_1A_n + \angle A_1B_2A_n + \cdots + \angle A_1B_{n-1}A_n = 180^\circ.$$

The pith of this problem is this lemma:

Lemma

Let $ABCD$ be an isosceles trapezoid, so that $\overline{AB} \parallel \overline{CD}$ and $BC = DA$. Let $S = \overline{AC} \cap \overline{BD}$, and suppose the angle bisector of $\angle BSC$ intersects $\overline{BC}$ at N. If P is the midpoint of $\overline{BC}$, then $ADPN$ is cyclic.

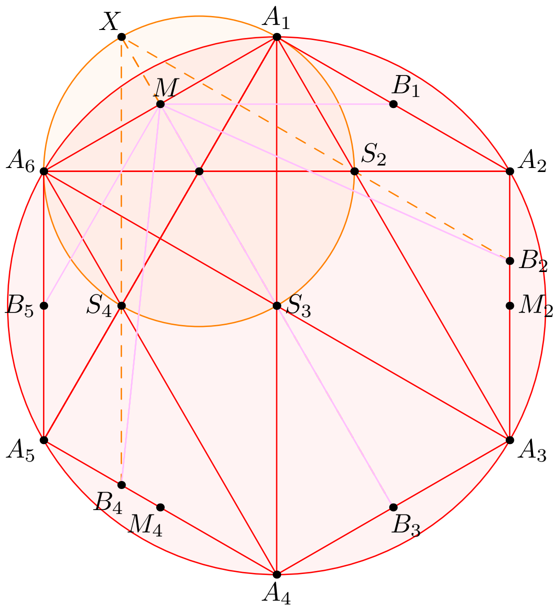
Proof. Let $T = \overline{AD} \cap \overline{BC}$ (if they are parallel the proof is trivial). Since $\overline{ST}$ bisects $\angle ASB$, by a well-known lemma, $-1 = (BC;TN)$. By the Midpoints of Harmonic Bundles Lemma, $TN \cdot TP = TB \cdot TC = TA \cdot TD$, and the desired result follows. □



Let M be the midpoint of $\overline{A_1A_n}$, and M_i the midpoint of $\overline{M_iM_{i+1}}$. Since $A_1A_n = A_iA_{i+1}$ and $A_1A_iA_{i+1}A_n$ is cyclic, $A_1A_iA_{i+1}A_n$ must be an isosceles trapezoid, whence by our lemma,

$$\sum_{i=1}^{n-1} \angle A_1B_iA_n = \sum_{i=1}^{n-1} \angle A_1M_iA_n = \sum_{i=1}^{n-1} \angle A_iMA_{i+1} = \angle A_1MA_n = 180^\circ,$$

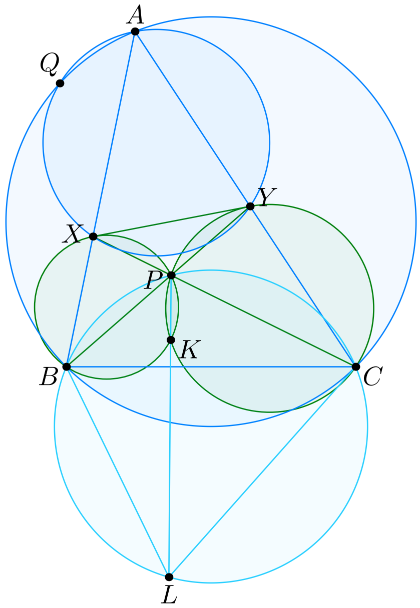
and we are done.



§2.12 Iran TST 2010/5

Problem 12 (Iran TST 2010/5)

Circles ω_1 and ω_2 intersect at points P and K. Line XY is tangent to ω_1 at X and ω_2 at Y. Let lines YP and XP meet ω_1 and ω_2, respectively, again at B and C, respectively, and let lines BX and CY meet at A. Prove that if the circumcircles of $\triangle ABC$ and $\triangle AXY$ meet again at Q, then $\angle QXA = \angle QKP$.



Note that K is the Miquel point of $BXCY$ and Q is the Miquel point of $BXYC$. Let the spiral similarity at Q sending $\overline{XY}$ to $\overline{BC}$ send K to L, so that $\triangle BLC \sim \triangle XKY$. Remark that

$$\begin{aligned} \measuredangle BLC = \measuredangle XKY &= \measuredangle XKP + \measuredangle XKP + \measuredangle PKY \\ &= \measuredangle YXP + \measuredangle PYX = \measuredangle YPX = \measuredangle BPC, \end{aligned}$$

so $BPCL$ is cyclic. Moreover,

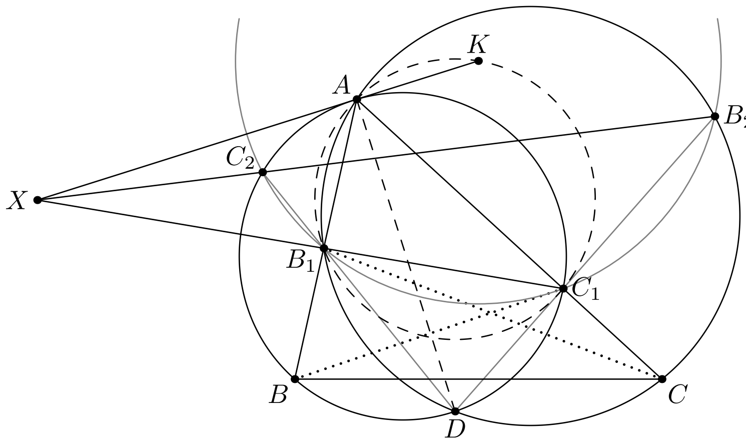
$$\measuredangle BPL = \measuredangle BCL = \measuredangle XYK = \measuredangle YCK = \measuredangle BXK = \measuredangle BPK,$$

whence L lies on $\overline{PK}$. It follows that $\measuredangle QKP = \measuredangle QKL = \measuredangle QXB = \measuredangle QXA$, as required.

§2.13 GOTEEM 2020/5 (Tovi Wen)

Problem 13 (GOTEEM 2020/5)

Let ABC be a triangle and let B_1 and C_1 be variable points on sides $\overline{BA}$ and $\overline{CA}$, respectively, such that $BB_1 = CC_1$. Let $B_2 \neq B_1$ denote the point on (ACB_1) such that $\overline{BC_1}$ is parallel to $\overline{B_1B_2}$, and let $C_2 \neq C_1$ denote the point on (ABC_1) such that $\overline{CB_1}$ is parallel to $\overline{C_1C_2}$. Prove that as B_1 and C_1 vary, the circumcircle of $\triangle AB_2C_2$ passes through a fixed point other than A.



Let K be the midpoint of arc BAC on the circumcircle of $\triangle ABC$, and let (ABC_1) and (ACB_1) intersect again at D. Denote $X = \overline{B_1B_2} \cap \overline{C_1C_2}$ and $Y = \overline{B_1C_1} \cap \overline{B_2C_2}$. I claim that K is the fixed point.

Since $KB = KC$, $BB_1 = CC_1$, and $\measuredangle KBB_1 = \measuredangle KCC_1$, $\triangle KBB_1 \cong \triangle KCC_1$, so K is the center of spiral similarity sending $\overline{BB_1}$ to $\overline{CC_1}$ and K lies on (AB_1C_1). Moreover, D is the center of spiral similarity sending $\overline{BB_1}$ to $\overline{C_1C}$. Since $BB_1 = CC_1$, we have $DB = DC_1$, so $\overline{AD}$ bisects $\angle BAC$.

Notice that

$$\measuredangle ADB_1 = \measuredangle ACB_1 = \measuredangle AC_1C_2 = \measuredangle ADC_2$$

and $\measuredangle ADC_1 = \measuredangle ADB_2$, so $D = \overline{B_1C_2} \cap \overline{B_2C_1}$. Furthermore

$$\measuredangle B_1B_2C_1 = \measuredangle B_1AD = \measuredangle DAC_1 = \measuredangle B_1DC,$$

so $B_1C_1B_2C_2$ is cyclic.

Finally A is the Miquel point of $B_1B_2C_1C_2$, so by a well-known property of complete quadrilaterals, A is the foot from X to $\overline{AD}$, id est $\angle XAD = 90^\circ$. It follows that $K \in \overline{AX}$, so

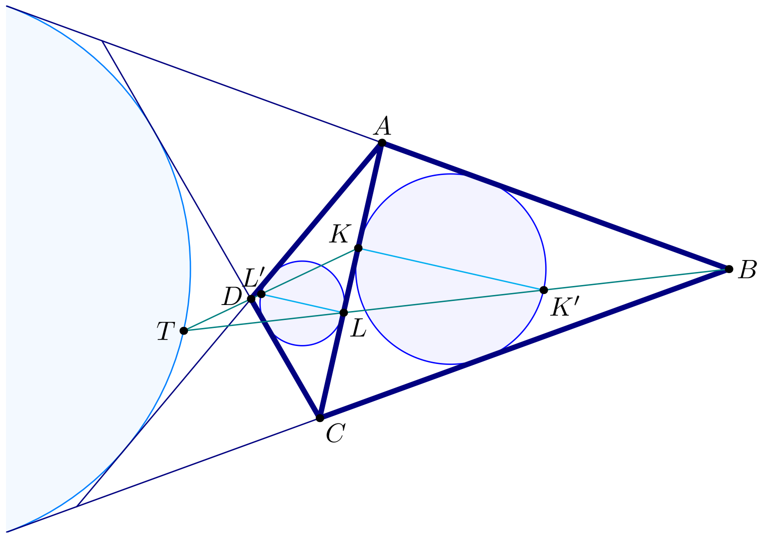
$$XB_2 \cdot XC_2 = XB_1 \cdot XC_1 = XA \cdot XK,$$

and we are done.

§2.14 IMO 2008/6 (Vladimir Shmarov)

Problem 14 (IMO 2008/6)

Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the extension of ray BA past A, to the extension of ray BC past C, to the line AD, and to the line CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.



Let ω touch $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ at T_1, T_2, T_3, T_4, respectively. Also let ω_1 and ω_2 touch $\overline{AC}$ at K and L, respectively, and let K' and L' be their respective antipodes. Denote by T the point on ω closest to $\overline{AC}$. Check that

$$\begin{aligned} AB + AD &= BT_1 - AT_1 + AT_4 - DT_4 = BT_2 - DT_3 \\ &= BT_2 - CT_2 + CT_3 - DT_3 = CB + CD. \end{aligned}$$

Note that

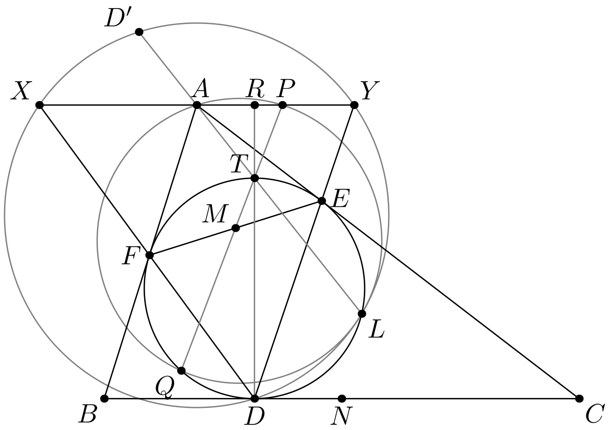
$$AK = \frac{AC + AB - CB}{2} = \frac{AC + CD - AD}{2} = CL,$$

so L is the B-extouch point of $\triangle ABC$. Analogously, K is the D-extouch point of $\triangle ADC$, so $K' \in \overline{BL}$ and $L' \in \overline{DK}$. The homothety at B between ω and ω_1 maps T to K', so $T \in \overline{BK'L}$. Similarly, the negative homothety at D between ω and ω_2 maps T to L', so $T \in \overline{DL'K}$. It follows that $T = \overline{K'L} \cap \overline{KL'}$. However, since $\overline{KK'} \parallel \overline{LL'}$, T is the center of homothety between $\overline{KK'}$ and $\overline{LL'}$, and since $\overline{KK'}$ and $\overline{LL'}$ are diameters of ω_1 and ω_2, T is the exsimilicenter of ω_1 and ω_2, which lies on ω, as desired.

§2.15 CAMO 2020/3 (Eric Shen)

Problem 15 (CAMO 2020/3)

Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Point M is the midpoint of $\overline{EF}$, and T is the point on ω such that $\overline{DT}$ is a diameter. Line MT meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines DF and DE intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.



Let $\overline{DT}$ intersect $\overline{AP}$ at R, and let $\overline{AT}$ intersect ω again at L.

Claim 1. T lies on $\overline{EX}$ and $\overline{FY}$, T is the orthocenter of $\triangle DXY$, and A is the midpoint of $\overline{XY}$.

Proof. Redefine $X = \overline{DF} \cap \overline{TE}$ and $Y = \overline{DE} \cap \overline{TF}$. Since $\angle DET = \angle DFT = 90^\circ$, T is the orthocenter of $\triangle DXY$. Thus, $\overline{DT} \perp \overline{XY}$, so $\overline{XY} \parallel \overline{BC}$.

By the Three Tangents lemma, the tangents to ω at E and F intersect at the midpoint of $\overline{XY}$; but this is A, thus recovering the original definitions of X and Y. $\square$

Claim 2. L lies on (DXY) and (APQ).

Proof. Since A is the midpoint of $\overline{XY}$ and T is the orthocenter of $\triangle DXY$, $\overline{AT}$ passes through D', the antipode of D on (DXY). Note that $\angle DLD' = \angle DLT = 90^\circ$, so L lies on (DXY).

Now $L \in (APQ)$ follows from $TA \cdot TL = TR \cdot TD = TP \cdot TQ$, thus proving the claim. $\square$

Finally, since $\overline{TM}$ is the T-symmedian of $\triangle TXY$ and L is the Miquel point of $XYEF$,

$$\frac{LX}{LY} = \frac{XF}{YE} = \frac{TX}{TY} = \left(\frac{PX}{PY}\right)^2.$$

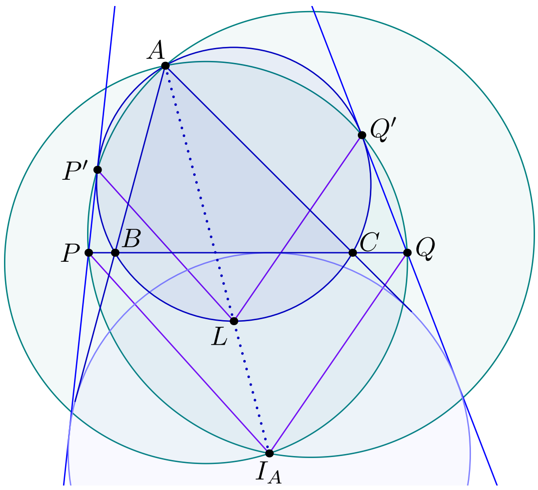
It follows that $\overline{LP}$ is a symmedian of $\triangle LXY$. Since $\overline{LA}$ and $\overline{LP}$ are isogonal, (LAP) and (DXY) are tangent, and we are done.

§2.16 Serbia 2017/6

Problem 16 (Serbia 2017/6)

Let ABC be a triangle and let the common external tangents to the circumcircle and the A-excircle intersect line BC at P and Q. Show that $\angle PAB = \angle CAQ$.

First solution, by elementary methods Let the tangents through P and Q touch (ABC) at P' and Q' respectively, and let L be the arc midpoint of BC and I_A the A-excenter.



Claim 1. $\overline{P'L} \parallel \overline{PI_A}$ (and thus $\overline{Q'L} \parallel \overline{QI_A}$).

Proof. Note that $\overline{LL} \parallel \overline{BC}$, so $\overline{P'L}$ is parallel to the external angle bisector of $\angle BPP'$. Since the excircle is tangent to lines BC and PP', we know $\overline{PI_A}$ externally bisects $\angle BPP'$, so $\overline{P'L} \parallel \overline{PI_A}$, as desired. □

Claim 2. $PP'AI_A$ is cyclic (thus so is $QQ'AI_A$).

Proof. This is obvious from the above claim: $\measuredangle PP'A = \measuredangle P'LA = \measuredangle PI_AA$. □

Finally, $\measuredangle PAI_A = \measuredangle PP'I_A = \measuredangle I_AQ'Q = \measuredangle I_AAQ$, so $\overline{AP}$ and $\overline{AQ}$ are isogonal with respect to $\angle BAC$, as desired.

Second solution, by Desargue involution Let S be the exsimilicenter and T the tangency point between (ABC) and the A-mixtillinear incircle. By Monge's theorem, A, T, S are collinear.

Let the excircle touch $\overline{BC}$ at D, and let $X = \overline{AST} \cap \overline{BC}$. By the dual of Desargue's involution theorem from S to $ABDC$ (with the excircle as the inconic), we have the involutive pairing $S(AD; BC; PQ)$. Projecting onto $\overline{BC}$ gives $(XD; BC; PQ)$, and perspectivity through A gives $A(TD; BC; PQ)$.

But $\overline{AT}$ and $\overline{AD}$ are isogonal in $\angle A$, so $\overline{AP}$ and $\overline{AQ}$ are also isogonal, and we are done.

§2.17 MOP 2019 + USA TST 2019/6 (Ankan Bhattacharya)

Problem 17 (MOP 2019 + USA TST 2019/6)

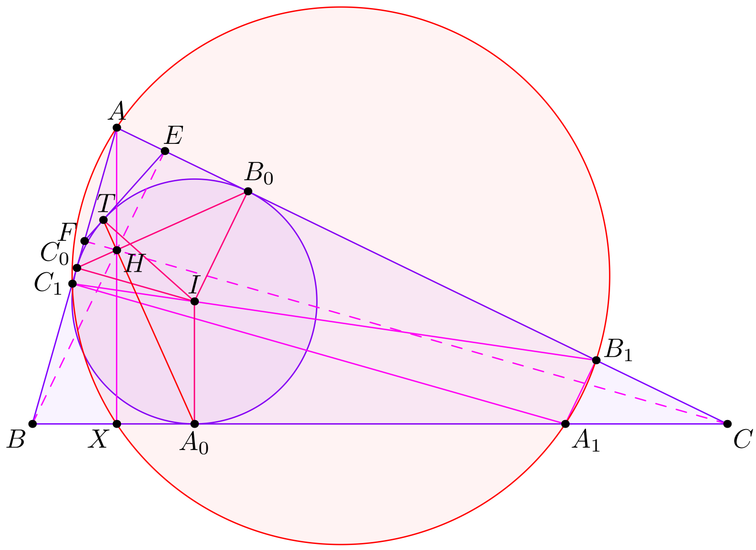
Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Denote by E and F the feet of the altitudes from B and C, respectively.

Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively. Prove that:

(a) (MOP 2019) If line EF is tangent to the incircle of $\triangle ABC$, then quadrilateral $AB_1A_1C_1$ is cyclic.

(b) (USA TST 2019/6) If quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.

Solution to part (a) It is easy to check that $\triangle ABC$ is acute, say by repeating the argument of JMO 2019/4.



Denote the intouch points by A_0, B_0, C_0, the orthocenter by H, the A-excenter by I_A, the Bevan point[1] by V, and the point where the incircle touches $\overline{EF}$ by T. By Brianchon's theorem on BCB_0EFC_0 and BA_0CETF, $\overline{B_0C_0}$ and $\overline{A_0T}$ intersect at H. Remark that $\overline{AH} \parallel \overline{IA_0}$. Since $BCEF$ is both cyclic and tangential,

$$\angle A_0HB_0 = \frac{\widehat{A_0B_0} + \widehat{TC_0}}{2} = \frac{(180^\circ - \angle B_0CA_0) + (180^\circ - \angle C_0FT)}{2} = 90^\circ,$$

whence $\overline{HA_0} \perp \overline{B_0C_0}$. However, $\overline{AI} \perp \overline{B_0C_0}$, so $\overline{AI} \parallel \overline{HA_0}$, and AHA_0I is a parallelogram.

Let R, r, r_A denote the circumradius, the inradius, and the A-exradius, respectively. Note that $2R\cos A = AH = IA_0 = r$. It is easy to see that $\triangle ABC \sim \triangle AEF$ with scale factor $\cos A$, and also r is the A-exradius of $\triangle AEF$, so $2R = r_A$. This implies that $I_AV = 2R = r_A = I_AA_1$. Since the excentral triangle is acute, $A_1 = V$, whence $\angle AB_1A_1 = \angle AC_1A_1 = 90^\circ$, and we are done.

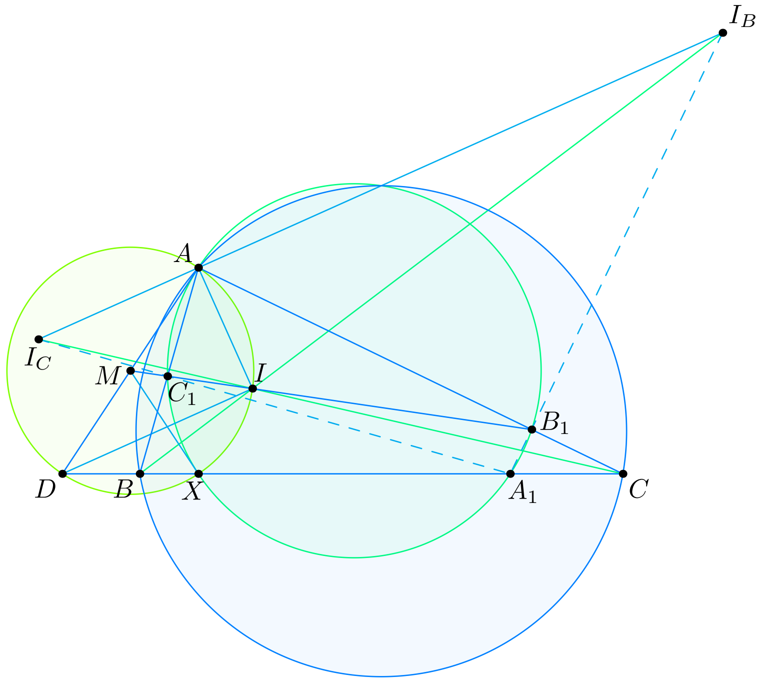
[1] the circumcenter of the excentral triangle

First solution to part (b), by harmonic bundles Let $V = \overline{I_B B_1} \cap \overline{I_C C_1}$ be the Bevan point of $\triangle ABC$. We know that $\overline{VA_1} \perp \overline{BC}$, $\overline{VB_1} \perp \overline{CA}$, $\overline{VC_1} \perp \overline{AB}$, so V lies on (AB_1C_1). If $A_1 \neq V$, then since $\overline{AV}$ is a diameter of (AB_1C_1), we have $\overline{AV} \parallel \overline{BC}$, which is absurd. Thus $V = A_1$, and $\overline{AA_1}$ is a diameter of (AB_1C_1).

By Pappus' theorem on $BACI_CA_1I_B$, we have $I \in \overline{B_1C_1}$. Denote by X the foot of the altitude from A, so that it lies on (AB_1C_1). Notice that

$$-1 = (A, \overline{BC} \cap \overline{I_B I_C}; I_B, I_C) \stackrel{A_1}{=} (AX; B_1C_1).$$

Let the tangents to (AB_1C_1) at A and X meet at M on $\overline{B_1C_1}$. Since I is the foot of the A-angle bisector of $\triangle AB_1C_1$ and AB_1XC_1 is harmonic, (AIX) is the A-Apollonius circle of $\overline{B_1C_1}$.



Since $\angle DIA = 90^\circ = \angle DXA$, $AIXD$ is cyclic. However M is the circumcenter of $\triangle AIX$, so M is the midpoint of $\overline{AD}$, and $MD^2 = MA^2 = MB_1 \cdot MC_1$. This completes the proof.

Second (official) solution to part (b), by angle chasing Let V be the antipode of A on (AB_1C_1). Let the incircle of $\triangle ABC$ touch $\overline{CA}$ and $\overline{AB}$ at E and F, respectively, and let J be the Miquel point of $BCEF$. Furthermore, let M be the midpoint of $\overline{AD}$.

We claim that $\overline{AA_1}$ is a diamter of $(AB_1A_1C_1)$. Note that V is the Bevan point of $\triangle ABC$, so $\overline{VA_1} \perp \overline{BC}$. Furthermore, if $V \neq A_1$, then $\overline{VA_1} \perp \overline{AA_1}$, which would require that $A \in \overline{BC}$, which is absurd.

By Pappus' theorem on $\overline{BA_1C}$ and $\overline{I_CAI_B}$, I lies on $\overline{B_1C_1}$, and by the Radical Axis theorem on (AI), (ABC), and (BIC), J lies on $\overline{AD}$. Since $\triangle JBF \sim \triangle JCE$,

$$\frac{AC_1}{AB_1} = \frac{JF}{JE} = \frac{JB}{JC},$$

and also $\measuredangle C_1AB_1 = \measuredangle BAC = \measuredangle BJC$, so $\triangle AC_1B_1 \sim \triangle JBC$.

This implies that

$$\measuredangle DAB = \measuredangle JAF = \measuredangle JEF = \measuredangle JCB = \measuredangle AB_1C_1,$$

so $\overline{AD}$ is tangent to (AB_1C_1). Moreover,

$$\measuredangle MIA = \measuredangle IAM = \measuredangle IAC_1 + \measuredangle C_1AM = \measuredangle B_1AI + \measuredangle C_1B_1A = \measuredangle B_1IA,$$

whence M lies on $\overline{B_1IC_1}$. Hence, $MD^2 = MA^2 = MB_1 \cdot MC_1$, and the desired result follows.

§2.18 ISL 2017 G7 (Kazakhstan)

Problem 18 (ISL 2017 G7)

A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_A, I_B, I_C, I_D be the incenters of the triangles DAB, ABC, BCD, CDA, respectively. Suppose that the common external tangents of the circumcircles of $\triangle AI_BI_D$ and $\triangle CI_BI_D$ meet at X, and the common external tangents of the circumcircles of $\triangle BI_AI_C$ and $\triangle DI_AI_C$ meet at Y. Prove that $\angle XIY = 90°$.

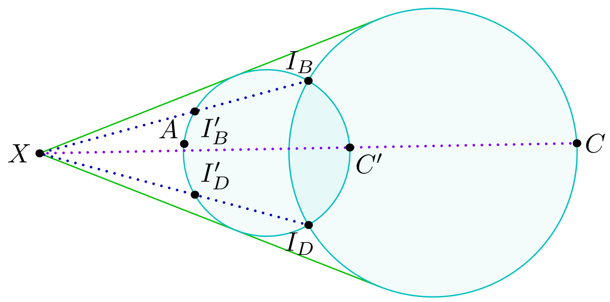
Lemma

Circles ω_A and ω_C, with the radius of ω_A less than the radius of ω_C, intersect at two points I_B and I_D. Point A is chosen on the circumference of ω_A but not in the interior of ω_C. Point C is chosen on the circumference of ω_C but not in the interior of ω_A. If the common external tangents of ω_A and ω_C intersect at X, then $\angle I_BXI_D = \angle I_BAI_D - \angle I_BCI_D$.

Proof. Denote by $\bullet'$ the homothety at X sending ω_C to ω_A. Then $X = \overline{I_BI'_B} \cap \overline{I_DI'_D}$, so with arcs taken with respect to ω_A,

$$\angle I_BXI_D = \frac{\widehat{I_BI_D} - \widehat{I'_BI'_D}}{2} = \angle I_BAI_D - \angle I'_BC'I'_D = \angle I_BAI_D - \angle I_BCI_D,$$

as claimed.



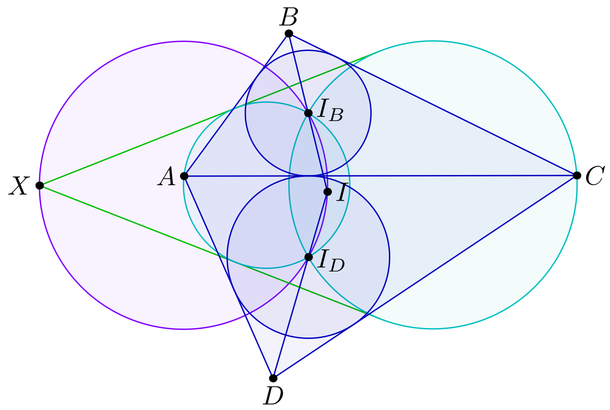
$\square$

Claim. I_BII_DX is cyclic.

Proof. Without loss of generality the radius of (AI_BI_D) is less than the radius of (CI_BI_D). By the lemma, $\angle I_BXI_D = \angle I_BAI_D - \angle I_BCI_D$. However

$$\angle I_BII_D = 360° - A - \frac{B}{2} - \frac{D}{2} = 180° - \frac{A}{2} + \frac{C}{2} = 180° - \angle I_BAI_D + \angle I_BCI_D,$$

so I_BII_DX is cyclic. $\square$



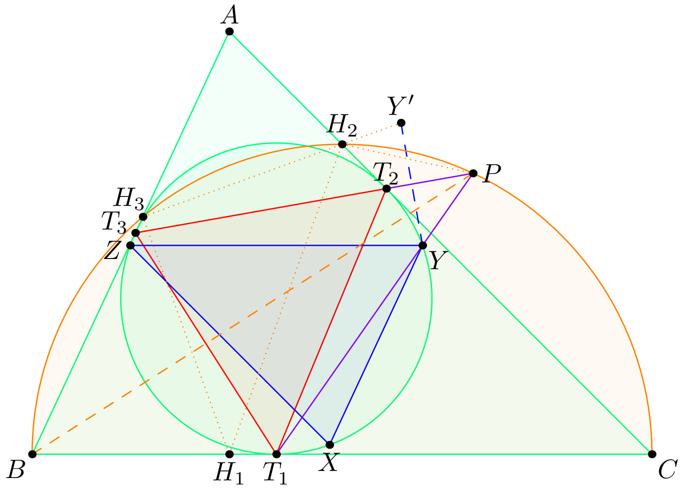
By symmetry, $XI_B = XI_D$, so $\overline{IX}$ bisects $\angle BID$. Analogously $\overline{IY}$ bisects $\angle AIC$. It is easy to check that $\angle AIB + \angle CID = 180^\circ$, so $\overline{AI}$ and $\overline{CI}$ are isogonal with respect to $\angle BID$. Hence $\overline{IX} \perp \overline{IY}$, as desired.

§2.19 IMO 2000/6

Problem 19 (IMO 2000/6)

In acute triangle ABC, let H_1, H_2, H_3 be the feet of the altitudes from A, B, C, respectively, and let T_1, T_2, T_3 be the points where the incircle touches $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Prove that the reflections of $\overline{H_1H_2}$, $\overline{H_2H_3}$, $\overline{H_3H_1}$ over $\overline{T_1T_2}$, $\overline{T_2T_3}$, $\overline{T_3T_1}$, respectively, are the sides of a triangle that is inscribed in the incircle of $\triangle ABC$.

Let Y lie on the incircle such that $\overline{T_2Y} \parallel \overline{T_3T_1}$. We will show that Y lies on the reflection of $\overline{H_2H_3}$ over $\overline{T_2T_3}$, which is sufficient by symmetry.



Let $P = \overline{T_1Y} \cap \overline{T_2T_3}$, which lies on $\overline{BI}$ by reflection, and let Y' be the reflection of Y over $\overline{T_2T_3}$. Then by the Iran Lemma, P lies on (BCH_2H_3). Since $\measuredangle H_2PB = \measuredangle H_2CB = \measuredangle T_2PY$, but $\overline{PB}$ bisects $\angle T_2PY$, $\overline{PH_2}$ bisects $\angle T_2PY'$, so we deduce by $PY' = PY = PT_2$ that PT_2H_2Y' is a kite. Now

$$\measuredangle PH_2Y' = \measuredangle T_2H_2P = \measuredangle CBP = \measuredangle PBH_3 = \measuredangle PH_2H_3,$$

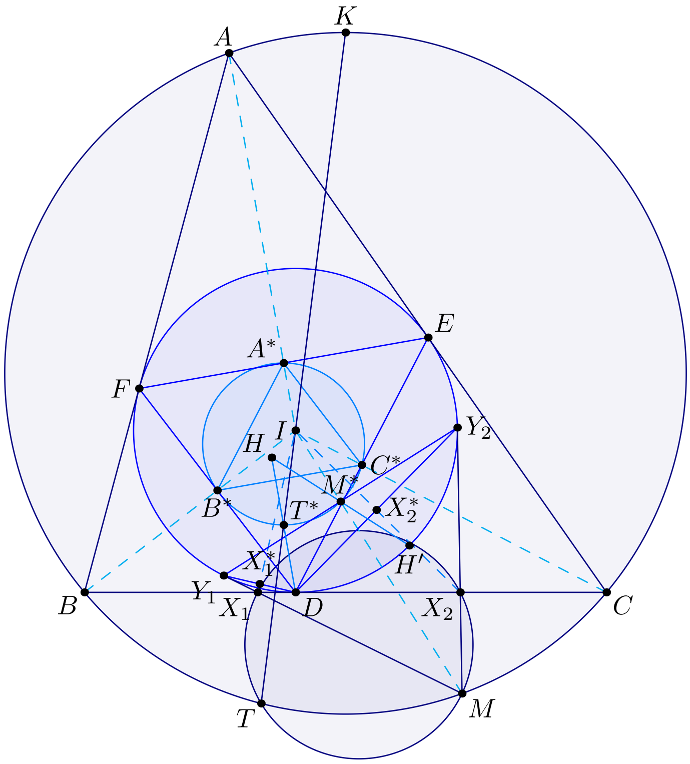
completing the proof.

§2.20 Taiwan TST 2014/3/3 (Cosmin Pohoatza)

Problem 20 (Taiwan TST 2014/3/3)

Let ABC be a triangle with circumcircle Γ and let M be an arbitrary point on Γ. Suppose that the tangents from M to the incircle of ABC intersect $\overline{BC}$ at two distinct points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 passes through the tangency point of the A-mixtilinear incircle with Γ.

First solution, by inversion Let the incircle touch $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F, respectively, and let $\overline{MX_1}$ and $\overline{MX_2}$ touch the incircle at Y_1 and Y_2, respectively. Denote by I, K, T, H, N the incenter of $\triangle ABC$, the midpoint of arc BAC on Γ, the point where the mixtilinear incircle of $\triangle ABC$ touches Γ, the orthocenter of $\triangle DEF$, and the nine-point center of $\triangle DEF$.



Invert through the incircle, denoting inversion by a star. For all points P, denote by ψ_P the homothety centered at P with scale factor 2, and let ϕ denote the homothety with scale factor -1 centered at M^*.

It is easy to check that $\triangle A^*B^*C^*$ and $\triangle DX_1^*X_2^*$ are the medial triangles of $\triangle DEF$ and $\triangle DY_2Y_1$, respectively. It is well-known that $\psi_H(A^*)$ is the antipode of D on (DEF), so $\overline{A^*N} \perp \overline{BC}$. Furthermore,

$$\measuredangle A^*M^*T^* = \measuredangle IM^*T^* = -\measuredangle ITM = -\measuredangle KTM = 90^\circ,$$

so T^* is the antipode of A^* with respect to Γ^*. Furthermore, $\psi_H(T^*) = D$. Let $D' = \psi_D(M^*)$. Since M^* lies on the nine-point circle of $\triangle DEF$, if $H' = \phi(H)$, then $H' \in (DEF)$. Hence, DY_2Y_1H' is cyclic. Since $\phi(D'Y_1Y_2H) = DY_2Y_1H'$, we have that $D'Y_1Y_2H$ is cyclic, and since $\psi_D(M^*X_1^*X_2^*T^*) = D'Y_1Y_2H$, we have that $M^*X_1^*X_2^*T^*$ is cyclic. It follows that $T \in (MX_1X_2)$, and we are done.

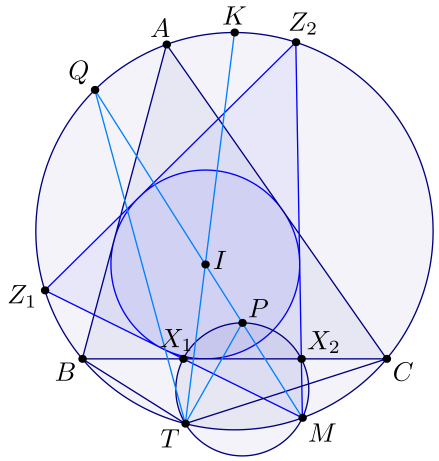
Second solution, by spiral similarity Let $\overline{MX_1}$ and $\overline{MX_2}$ intersect Γ again at Z_1 and Z_2, and let the circumcircle of $\triangle MX_1X_2$ intersect Γ again at T. Also let P and Q be the midpoints of arcs X_1X_2 and Z_1Z_2 on (MX_1X_2) and Γ, respectively.

By Poncelet's Porism, $\overline{Z_1Z_2}$ is tangent to the incircle. Check that the incircle of $\triangle ABC$ serves as the incircle of $\triangle MZ_1Z_2$ and the M-excircle of $\triangle MX_1X_2$, and furthermore T is the center of spiral similarity sending $\overline{X_1X_2}$ to $\overline{Z_1Z_2}$.

This spiral similarity clearly sends P to Q, so we obtain from the Incenter-Excenter Lemma that

$$\frac{TP}{TQ} = \frac{PX_1}{QZ_1} = \frac{PI}{QI},$$

whence $\overline{TI}$ bisects $\angle PTQ$.



It is obvious that $\overline{TP}$ bisects $\angle X_1TX_2$ and $\overline{TQ}$ bisects $\angle Z_1TZ_2$. Since $\measuredangle BZ_1T = \measuredangle BCT = \measuredangle X_2CT$ and

$$\measuredangle TBZ_1 = \measuredangle TZ_2Z_1 = \measuredangle TX_2X_1 = \measuredangle TX_2C,$$

we determine that $\triangle TBZ_1 \sim \triangle TX_2C$. Finally,

$$\begin{aligned}\measuredangle BTI &= \measuredangle BTZ_1 + \measuredangle Z_1TQ + \measuredangle QTI \\ &= \measuredangle X_2TC + \measuredangle PTX_2 + \measuredangle ITP \\ &= \measuredangle ITC,\end{aligned}$$

so $\overline{TI}$ passes through the midpoint of $\widehat{BAC}$, and we are done.

Third solution, by DDIT Let I, T, D, A', M' denote the incenter of $\triangle ABC$, the A-mixtilinear touch point, the A-intouch point, the point on (ABC) such that $\overline{AA'} \parallel \overline{BC}$, and the intersection of $\overline{AM}$ and $\overline{BC}$.

By the 3-point case of the Dual of Desargues' Involution Theorem, there exists an involution swapping $(\overline{MA}, \overline{MD})$, $(\overline{MB}, \overline{MC})$, and $(\overline{MX_1}, \overline{MX_2})$. Projecting onto $\overline{BC}$, there is an involution swapping (M', D), (B, C), and (X_1, X_2). Call the center of this involution O, so that $OM' \cdot OD = OB \cdot OC = OX_1 \cdot OX_2$. It follows that O has equal power to (ABC), (DMM'), and (MX_1X_2), so the three circles are coaxial.

It therefore suffices to show that T lies on (DMM'). However, it is well-known that $\angle BTA = \angle DTC$, so A' lies on line DT. Thus,

$$\measuredangle TMM' = \measuredangle TMA = \measuredangle TA'A = \measuredangle DA'A = \measuredangle A'DC = \measuredangle TDM',$$

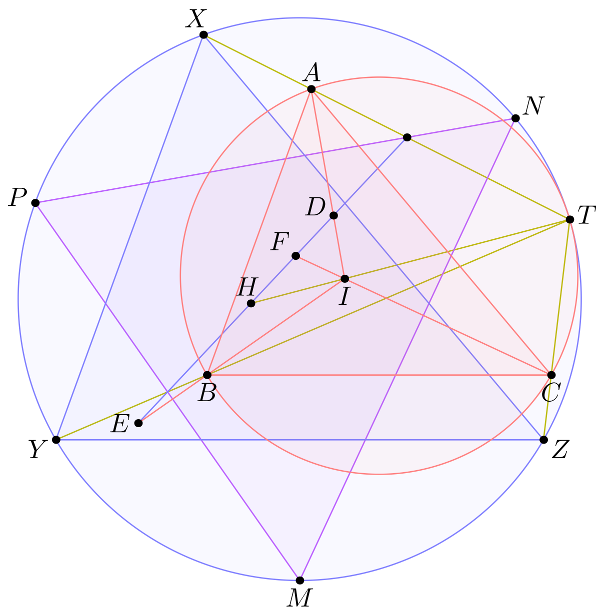
as required.

§2.21 ISL 2018 G5 (Denmark)

Problem 21 (ISL 2018 G5)

Let ABC be a triangle with circumcircle ω and incenter I. A line ℓ meets the lines AI, BI, and CI at points D, E, and F, respectively, all distinct from A, B, C and I. Prove that the circumcircle of the triangle determined by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ is tangent to ω.

First solution, by angle chasing Let $\ell = \overline{DEF}$ and ℓ_A, ℓ_B, and ℓ_C denote its reflections across the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively.



Claim. Lines ℓ_A, ℓ_B, ℓ_C concur at a point T on ω.

Proof. Check that

$$\measuredangle(\ell_B, \ell_C) = \measuredangle(\ell_B, \ell) + \measuredangle(\ell, \ell_C) = 2\measuredangle(\ell_B, \overline{BI}) + 2\measuredangle(\overline{CI}, \ell_C) = 2\measuredangle(\ell_B, \ell_C) + 2\measuredangle CIB,$$

whence $\measuredangle(\ell_B, \ell_C) = 2\measuredangle BIC = \measuredangle BAC$, as desired. $\square$

Let $H = \overline{TI} \cap \ell$. A homothety at T maps I to H and $\triangle ABC$ to some triangle $\triangle XYZ$ whose incenter is H such that (ABC) and (XYZ) are tangent at T.

Recall that by design, the perpendicular bisector of $\overline{XH}$ coincides with the perpendicular bisector of $\overline{AD}$, so the vertices of the triangle formed by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are the arc midpoints of $\triangle XYZ$, which completes the proof.

Second solution, by Simson Lines (Pitchayut Saengrungkongka) Let the midpoints of arcs BC, CA, AB not containing A, B, C be M_A, M_B, M_C respectively, and let lines through D, E, F perpendicular to $\overline{AI}$, $\overline{BI}$, $\overline{CI}$, respectively form a triangle $A_1B_1C_1$. Also let XYZ be the triangle described in the problem. Since $\triangle M_AM_BM_C$ and $\triangle XYZ$ are homothetic, $\overline{XM_A}$, $\overline{YM_B}$, $\overline{ZM_C}$ concur at a point T.

Claim. $IA_1B_1C_1$ and $TM_AM_BM_C$ are homothetic.

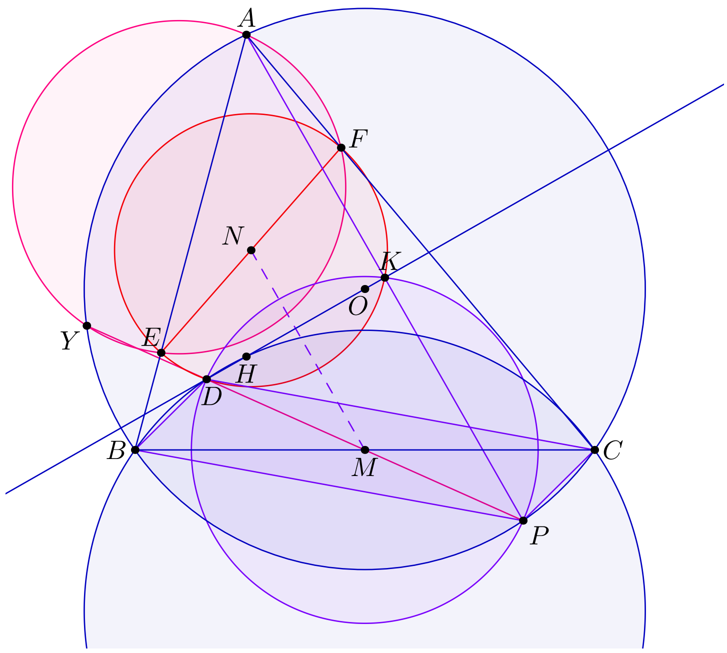
Proof. Since $\triangle A_1B_1C_1$ and $\triangle M_AM_BM_C$ are homothetic it suffices to show that $\overline{IA_1} \parallel \overline{M_AX}$. Let I_A be the A-excenter. Then $\overline{M_AX}$ is the I_A-midsegment of $\triangle I_AIA_1$, since the projections of I_A, A_1, X onto $\overline{BI}$ are B, the midpoint of $\overline{BE}$, and E. Thus the claim is true. $\square$

To finish, note that $\overline{DEF}$ is the Simson Line from I to $\triangle A_1B_1C_1$, so $IA_1B_1C_1$ and thus $TM_AM_BM_C$ is cyclic. By homothety, (ABC) and (XYZ) are tangent at T, so we are done.

§2.22 Iran TST 2017/3/6 (Iman Maghsoudi)

Problem 22 (Iran TST 2017/3/6)

Let ABC be a triangle with circumcenter O and orthocenter H. Point P is the reflection of A with respect to $\overline{OH}$. Assume that P is not on the same side of $\overline{BC}$ as A. Points E and F lie on $\overline{AB}$ and $\overline{AC}$ respectively such that $BE = PC$ and $CF = PB$. Let K be the intersection of $\overline{AP}$ and $\overline{OH}$. Prove that $\angle EKF = 90^\circ$.



Let M be the midpoint of $\overline{BC}$, N the midpoint of $\overline{EF}$, D the point such that $BPCD$ is a parallelogram, and Y the second intersection of $\overline{PMD}$ with (ABC).

Claim 1. D lies on $\overline{OH}$.

Proof. By reflection through M, D lies on (BHC). Thus

$$\measuredangle BHD = \measuredangle BCD = \measuredangle CBP = \measuredangle CAP = 90^\circ + \measuredangle(\overline{AC} + \overline{OH}) = \measuredangle BHO,$$

as desired. □

Claim 2. $\angle EDF = 90^\circ$.

Proof. Since $BE = PC = BD$ and $CF = PB = CD$,

$$\begin{aligned}\measuredangle EDF &= \measuredangle EDB + \measuredangle BDC + \measuredangle CDF = \measuredangle BED + \measuredangle CPB + \measuredangle DFC \\ &= \measuredangle AED + \measuredangle FAE + \measuredangle DFA = \measuredangle FDE.\end{aligned}$$

Since $D \notin \overline{EF}$, it follows that $\angle EDF = 90^\circ$. □

Claim 3. Y is the Miquel point of $BCFE$; in particular, $Y \in (AEF)$.

Proof. Since $M \in \overline{YP}$, we have

$$\frac{YB}{YC} = \frac{PC}{PB} = \frac{BE}{CF},$$

whence $\triangle YBE \sim \triangle YCF$, as desired. □

Claim 4. $\overline{MN} \perp \overline{OH}$.

Proof. A spiral similarity at Y sends $\overline{BC} \cup M$ to $\overline{EF} \cup N$, so

$$\measuredangle YMN = \measuredangle YBE = \measuredangle YBA = \measuredangle YPA = \measuredangle(\overline{YM}, \overline{AP}),$$

it follows that $\overline{MN} \parallel \overline{AP}$, which is sufficient. $\square$

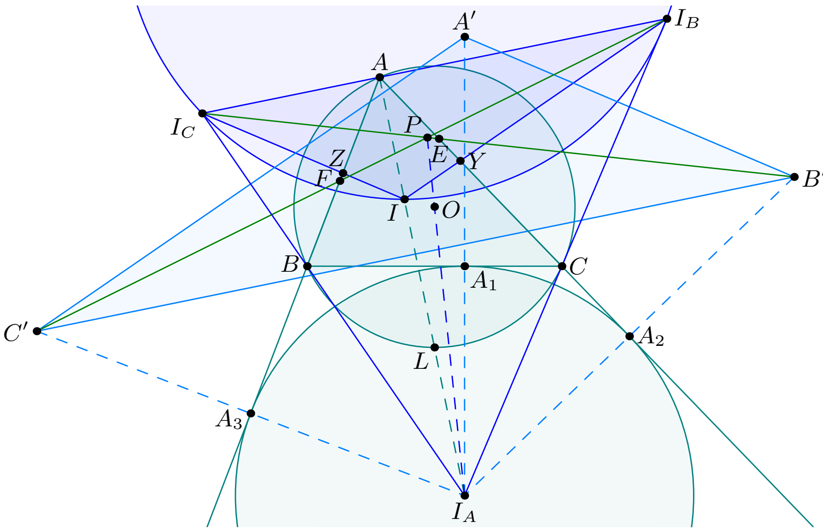
Finally since $\angle DKP = 90^\circ$, M is the center of (DKP), whence D and K are reflections across $\overline{MN}$. It follows that $K \in (DEF)$, so $\angle BKC = \angle BDC = 90^\circ$, and we are done.

§2.23 USAMO 2016/3 (Evan Chen, Telv Cohl)

Problem 23 (USAMO 2016/3)

Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines I_BF and I_CE meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.



Let I be the incenter and I_A the A-excenter of $\triangle ABC$. Denote by A', B', C' the reflections of I_A across $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively, and A_1, B_2, C_3 the projections. The key claim is this:

Claim. Points D, I, A' are collinear, and so are B', E, I_C and C', F, I_B.

Proof. It is well-known that $\overline{IA_1}$ bisects $\overline{AD}$. Since A, I, I_A are collinear, so are D, I, A'. Now, we will show that $\overline{I_CA_2}$ bisects $\overline{BE}$, whence the desired collinearity (B', E, I_C) follows from a similar argument to above.

Let C_2 be the projection of I_C onto $\overline{AC}$, so that $\overline{I_AA_2} \parallel \overline{I_CC_2}$, and set $K = \overline{I_CA_2} \cap \overline{I_AC_2}$. By the homothety centered at B sending the A- to C-excircle,

$$\frac{BI_A}{BI_C} = \frac{I_AA_2}{I_CC_2} = \frac{KA_2}{KI_C},$$

from which $\overline{BK} \parallel \overline{I_AA_2}$, and thus $\overline{BK} \perp \overline{AA_2}$. Now, if $S = \overline{AC} \cap \overline{I_AI_C}$ we may conclude by properties of trapezoid $I_AA_2C_2I_C$ that K is the midpoint of $\overline{BE}$.

By symmetry, our claim has been proven.[2] □

Since $I_AB' = 2I_AB_0 = 2I_AA_0 = I_AA'$ and $CB' = CI_A = CA'$, $\overline{A'B'} \perp \overline{I_AC}$, whence $\overline{A'B'} \parallel \overline{II_C}$. Similarly $\overline{A'C'} \parallel \overline{II_B}$ and $\overline{B'C'} \parallel \overline{I_BI_C}$, thus $\triangle A'B'C'$ and $\triangle II_BI_C$ are homothetic with center P. It follows that P lies on line OI_A.

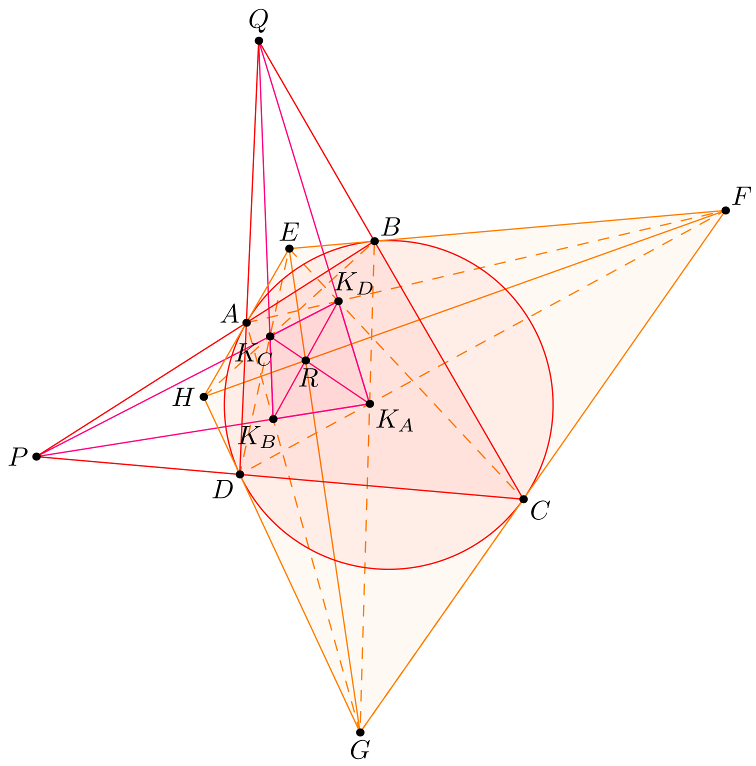
However, $\overline{OI_A}$ is the Euler line and $\overline{YZ}$ the orthic axis of $\triangle II_BI_C$. It is well-known that they are perpendicular, whence $\overline{PO} \perp \overline{YZ}$. This completes the proof.

[2] An alternate proof is to notice that $-1 = (B, \overline{AC} \cap \overline{I_AI_C}; I_C, I_A) \stackrel{B_0}{=} (B, E; \overline{BE} \cap \overline{A_2I_C}, \infty_{\perp AC})$.

§2.24 Sharygin 2014/10.8 (Nikolai Beluhov)

Problem 24 (Sharygin 2014/10.8)

Let $ABCD$ be a cyclic quadrilateral. Prove that if the symmedian points of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, $\triangle DAB$ are concyclic, then quadrilateral $ABCD$ has two parallel sides.



Let K_A, K_B, K_C, K_D be the symmedian points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, $\triangle ABC$, and let $E = \overline{AA} \cap \overline{BB}$, $F = \overline{BB} \cap \overline{CC}$, $G = \overline{CC} \cap \overline{DD}$, $H = \overline{DD} \cap \overline{AA}$, $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{AD} \cap \overline{BC}$, $R = \overline{AC} \cap \overline{BD}$. Assume for contradiction that $ABCD$ has no two sides parallel (i.e. P and Q are not infinity) but $K_AK_BK_CK_D$ is cyclic. The key claim is this:

Claim. $P = \overline{K_AK_B} \cap \overline{K_CK_D}$, $Q = \overline{K_AK_D} \cap \overline{K_BK_C}$, $R = \overline{K_AK_C} \cap \overline{K_BK_D}$.

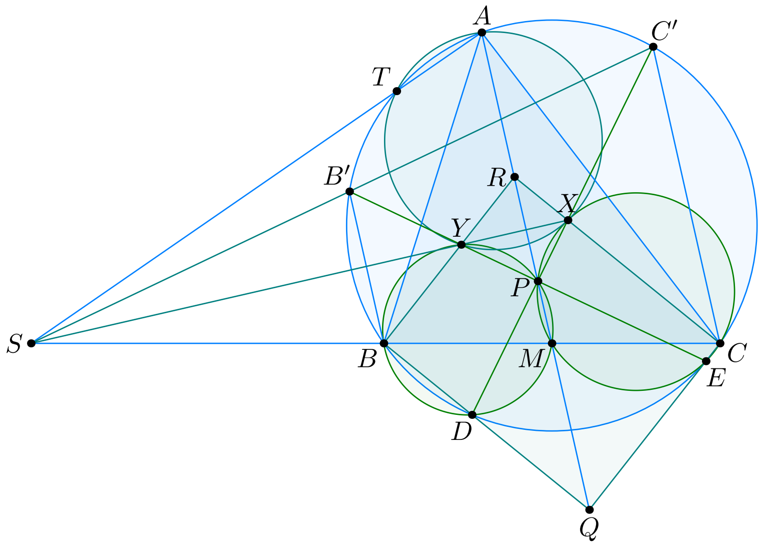
Proof. By design, $K_A = \overline{BG} \cap \overline{DF}$ and $K_C = \overline{BH} \cap \overline{DE}$. Applying Brianchon theorem on hexagons $EBFGDH$ and $EFCGHA$ shows that $R = \overline{EG} \cap \overline{RH}$, so by Pappus theorem on $EDFHBG$, points K_A, R, K_C are collinear. Symmetrically $R = \overline{K_AK_C} \cap \overline{K_BK_D}$, and analogous arguments show that $P = \overline{K_AK_B} \cap \overline{K_CK_D}$ and $Q = \overline{K_AK_D} \cap \overline{K_BK_C}$, as desired. □

This implies that $(K_AK_BK_CK_D)$ is the polar circle of $\triangle PQR$; however so is $(ABCD)$. Since the polar circle is unique, $(ABCD)$ and $(K_AK_BK_CK_D)$ coincide, absurd.

§2.25 APMO 2019/3 (Mexico)

Problem 25 (APMO 2019/3)

Let ABC be a scalene triangle with circumcircle Γ, and let M be the midpoint of $\overline{BC}$. A variable point P is selected on $\overline{AM}$. The circumcircles of triangles BPM and CPM intersect Γ again at points D and E, respectively. Lines DP and EP intersect the circumcircles of triangles CPM and BPM again at X and Y, respectively. Prove that as P varies, the circumcircle of $\triangle AXY$ passes through a fixed point T distinct from A.



Let B' and C' lie on Γ with $\overline{AM} \parallel \overline{BB'} \parallel \overline{CC'}$. Denote $S = \overline{BC} \cap \overline{B'C'}$, and let $\overline{AS}$ intersect Γ again at T. I claim that T is the fixed point.

Claim 1. $BCXY$ is cyclic.

Proof. By radical axes on Γ, (BPM), (CPM), we know $\overline{AM}$, $\overline{BD}$, $\overline{CE}$ concur at a point Q. Let R be the point such that $BQCR$ is a parallelogram. By Reim's theorem on (BPM) and (CPM), $\overline{BY} \parallel \overline{CE}$, so Y lies on $\overline{BR}$. Similarly X lies on $\overline{CR}$, so $\overline{AM}$, $\overline{BY}$, $\overline{CX}$ concur at R.

From here it is easy to check that $RB \cdot RY = RP \cdot RM = RC \cdot RX$, as desired. □

Claim 2. $B'C'XY$ is cyclic.

Proof. By Reim's theorem on Γ and (CPM), we have E, P, B' collinear. Similarly D, P, C' collinear.

Since $\measuredangle CXY = \measuredangle CBY = \measuredangle BCE = \measuredangle BDE$, we have $\overline{XY} \parallel \overline{DE}$, so $B'C'XY$ is cyclic by Reim's theorem. □

By the Radical Axis Theorem on Γ, $(BCXY)$, $(B'C'XY)$, point S lies on $\overline{XY}$. To finish, note that $SA \cdot ST = SB \cdot SC = SX \cdot SY$, whence T lies on (AXY). Since T is fixed, we are done.

§2.26 IMO 2018/6 (Poland)

Problem 26 (IMO 2018/6)

A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

First solution, by inversion We first require the following two lemmas.

Lemma 1

If two quadrilaterals have the same angles and both obey $AB \cdot CD = BC \cdot DA$, then they are similar.

Proof. Omitted. □

Lemma 2

If point S in quadrilateral $ABCD$ has a isogonal conjugate S^*, then $\angle BSA + \angle DSC = 180^\circ$.

Proof. Let $P = \overline{AB} \cap \overline{CD}$ and $Q = \overline{AD} \cap \overline{BC}$, and denote by W, X, Y, Z the projections of S onto $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively. Note that S and S^* are isogonal conjugates with respect to the four triangles $\triangle PAD$, $\triangle PBC$, $\triangle QAB$, and $\triangle QDC$. Since the center of the pedal circle of S is the midpoint of $\overline{SS^*}$, points W, X, Y, Z lie on the pedal circle of S.

Now, all that remains is an angle chase:

$$\begin{aligned}\measuredangle BSA + \measuredangle DSC &= \measuredangle BSW + \measuredangle WSA + \measuredangle DSY + \measuredangle YSC \\ &= \measuredangle BXW + \measuredangle WZA + \measuredangle DZY + \measuredangle YXC \\ &= \measuredangle WZY + \measuredangle YXW = 0^\circ,\end{aligned}$$

as desired. □

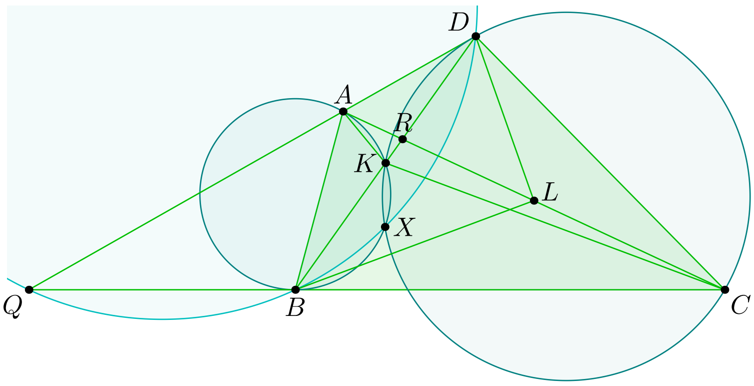
Now, invert about X with arbitrary radius r, denoting the inverse of T by T'. Notice that $\measuredangle XB'A' = -\measuredangle XAB = -\measuredangle XCD = \measuredangle XD'C'$, and similarly $\measuredangle XC'B' = \measuredangle XA'D'$. Furthermore by the Inversion Distance Formula,

$$A'B' \cdot C'D' = \frac{r^2 \cdot AB}{XA \cdot XB} \cdot \frac{r^2 \cdot CD}{XC \cdot XD} = \frac{r^2 \cdot BC}{XB \cdot XC} \cdot \frac{r^2 \cdot DA}{XD \cdot XA} = B'C' \cdot D'A'.$$

We can also check that

$$\measuredangle D'A'B' = \measuredangle D'A'X + \measuredangle XA'B' = \measuredangle XDA + \measuredangle ABX = \measuredangle XBC + \measuredangle ABX = \measuredangle ABC,$$

and analogously we find by Lemma 1 that $D'A'B'C' \sim ABCD$. Transforming $D'A'B'C'$ back to $ABCD$, X is mapped to its isogonal conjugate, so by Lemma 2, $\angle BXA + \angle DXC = 180^\circ$, and we are done.



Second solution, by angle chasing Let $Q = \overline{AD} \cap \overline{BC}$. Since $AB/AD = CB/CD$, there exists a point E on $\overline{BD}$ such that $\overline{AE}$ bisects $\angle DAB$ and $\overline{CE}$ bisects $\angle BCD$. Thus there exists a point K on $\overline{BD}$ with $\measuredangle CAB = \measuredangle DAK$ and $\measuredangle BCA = \measuredangle KCD$. Let the circumcircles of $\triangle AKB$ and $\triangle CKD$ intersect at X. I claim that X is the desired point. First, we prove a key claim.

Claim. $\overline{BD}$ bisects $\angle AKC$.

Proof. Notice that

$$\frac{KA}{KD} = \frac{\sin \angle BDA}{\sin \angle KAD} = \frac{\sin \angle BDA}{\sin \angle BAC} \quad \text{and} \quad \frac{KC}{KD} = \frac{\sin \angle BDC}{\sin \angle KCD} = \frac{\sin \angle BDC}{\sin \angle BCA}.$$

By the ratio lemma,

$$\frac{KA}{KC} = \frac{\sin \angle BDA}{\sin \angle BDC} \cdot \frac{\sin \angle BCA}{\sin \angle BAC} = \frac{RA}{RC} \cdot \frac{DC}{DA} \cdot \frac{BA}{BC} = \frac{RA}{RC},$$

and the desired result readily follows. □

Notice that by the claim, $\measuredangle BXA + \measuredangle DXC = \measuredangle DXA + \measuredangle BXC = \measuredangle DKA + \measuredangle BKC = 0^\circ$, so it is sufficient to show that $\measuredangle XBC = \measuredangle XDA$ (and the other case follows analogously). But

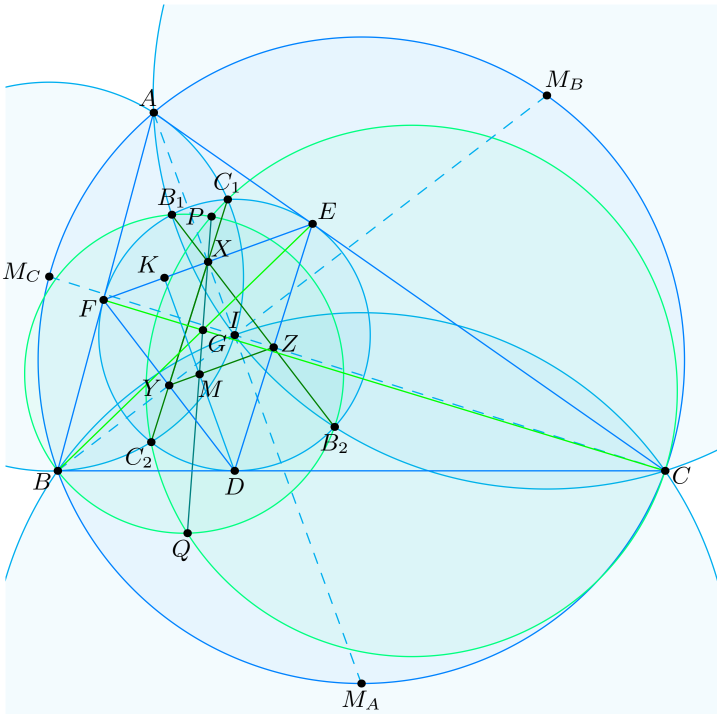
$$\measuredangle BXD = \measuredangle BXK + \measuredangle KXD = \measuredangle BAK + \measuredangle KCD = \measuredangle CAD + \measuredangle BCA = \measuredangle CQA,$$

so $BQDX$ is cyclic and $\measuredangle XBC = \measuredangle XBQ = \measuredangle XDQ = \measuredangle XDA$, as desired.

§2.27 TSTST 2016/6 (Danielle Wang)

Problem 27 (TSTST 2016/6)

Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.



Let $\overline{AI}$, $\overline{BI}$, $\overline{CI}$ intersect (ABC) again at M_A, M_B, M_C, respectively, and let X, Y, Z be the midpoints of $\overline{EF}$, $\overline{FD}$, $\overline{DE}$, respectively. Denote by G the Gergonne point of $\triangle ABC$, and let (BB_1B_2) and (CC_1C_2) intersect at P and Q.

By the Incenter-Excenter Lemma, M_A, M_B, M_C are the circumcenters of triangles BIC, CIA, and AIB, respectively.

Claim 1. $\overline{AI}$, $\overline{EF}$, $\overline{B_1B_2}$, $\overline{C_1C_2}$, and $\overline{PQ}$ concur at X.

Proof. Clearly $\overline{AI}$ and $\overline{EF}$ intersect at X. By the Radical Axis Theorem on $(AEIF)$, (DEF), and (AIC), $\overline{B_1B_2}$ passes through X, and similarly $C \in \overline{C_1C_2}$. Finally, $XB_1 \cdot XB_2 = XC_1 \cdot XC_2$, so X lies on the radical axis of (BB_1B_2) and (CC_1C_2), which is $\overline{PQ}$, as desired. $\square$

Claim 2. $\overline{B_1B_2}$ and $\overline{C_1C_2}$ are the E- and F-midsegments of $\triangle DEF$.

Proof. Note that $\overline{B_1B_2}$ is the radical axis of (AIC) and (DEF), so $\overline{B_1B_2}$ is perpendicular to $\overline{BIM_B}$. However, so is $\overline{FD}$, so $\overline{FD} \parallel \overline{B_1B_2}$. By Claim 1, $X \in \overline{B_1B_2}$, so the claim is proven. $\square$

Claim 3. G lies on $\overline{PQ}$.

Proof. First, since $B_1Z \cdot B_2Z = IZ \cdot CZ$, $\overline{BZ}$ is the radical axis of (BB_1B_2) and (BIC). By the Radical Axis Theorem on (BB_1B_2), (CC_1C_2), and (BIC), $\overline{BZ} \cap \overline{CY}$ lies on $\overline{PQ}$.

Now, by Cevian Nest on $\triangle GBC$, $\triangle DEF$, and $\triangle XYZ$, $\overline{GX}$ lies on $\overline{PQ}$, whence G lies on $\overline{PQ}$, as desired. $\square$

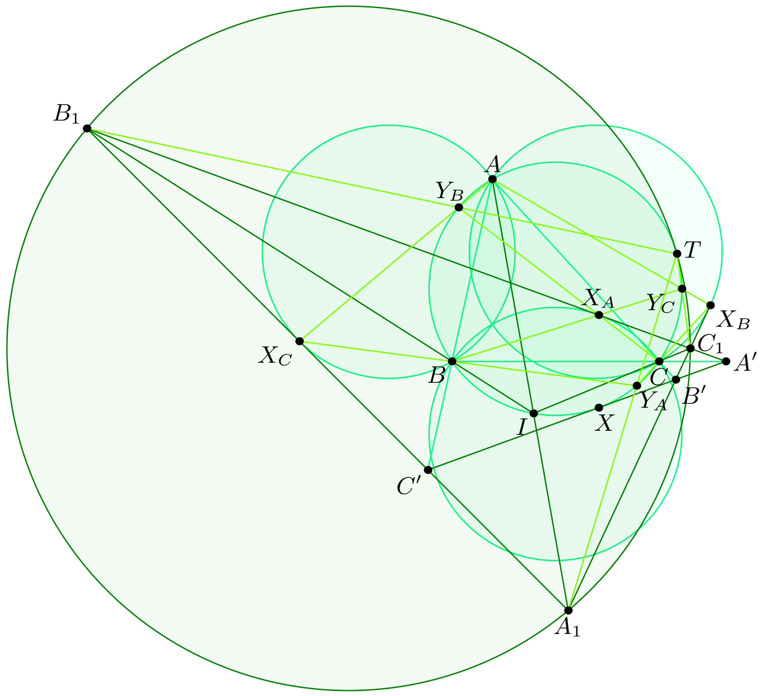
Since G is the symmedian point of $\triangle DEF$, it is well-known that M, G, X are collinear, whence M lies on $\overline{PQ}$, as desired.

§2.28 IMO 2011/6 (Japan)

Problem 28 (IMO 2011/6)

Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b, and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b, and ℓ_c is tangent to the circle Γ.

Let ℓ touch Γ at X. Make the following definitions with cyclic variations defined similarly as well: Denote $A_1 = \ell_b \cap \ell_c$ and $A' = \ell \cap \ell_a$. Let X_A be the reflection of X over $\overline{BC}$ and $Y_A = \overline{BX_C} \cap \overline{CX_B}$.



Since

$$\measuredangle AB'C' = \measuredangle AB'X = \measuredangle X_B B'A = \measuredangle A_1 B'A,$$

and similarly $\measuredangle B'C'A = \measuredangle AC'A_1$, A is either the incenter of A_1-excenter of $\triangle A_1B'C'$. Thus, $\overline{A_1A}$ bisects $\angle C_1AB_1$, so by symmetry, $\overline{A_1A}$, $\overline{B_1B}$, $\overline{C_1C}$ concur at the incenter I of $\triangle A_1B_1C_1$. Now, remark that

$$\measuredangle BAC = 90^\circ - \measuredangle B'A_1I = \measuredangle B_1IC = \measuredangle BIC,$$

whence $I \in (ABC)$. Since ℓ is tangent to (ABC), ℓ_a is tangent to (X_ABC), so

$$\measuredangle ABY_A = \measuredangle ABX_C = \measuredangle XBA = \measuredangle XCA = \measuredangle ACX_B = \measuredangle ACY_A,$$

so $Y_A \in (ABC)$. By symmetry, $Y_B, Y_C \in (ABC)$ as well. Then,

$$\measuredangle Y_BY_CB = \measuredangle Y_BCB = \measuredangle X_ACB = \measuredangle B_1X_AB,$$

so $\overline{Y_BY_C} \parallel \ell_a$, from which by symmetry, $\triangle A_1B_1C_1$ and $\triangle Y_AY_BY_C$ are homothetic, say with center T. However, since B_1, C_1, X_A are collinear, by the converse of Pascal's Theorem on Y_BTY_CBIC, $T \in \Gamma$, so Γ and $(A_1B_1C_1)$ are tangent at T, as desired.

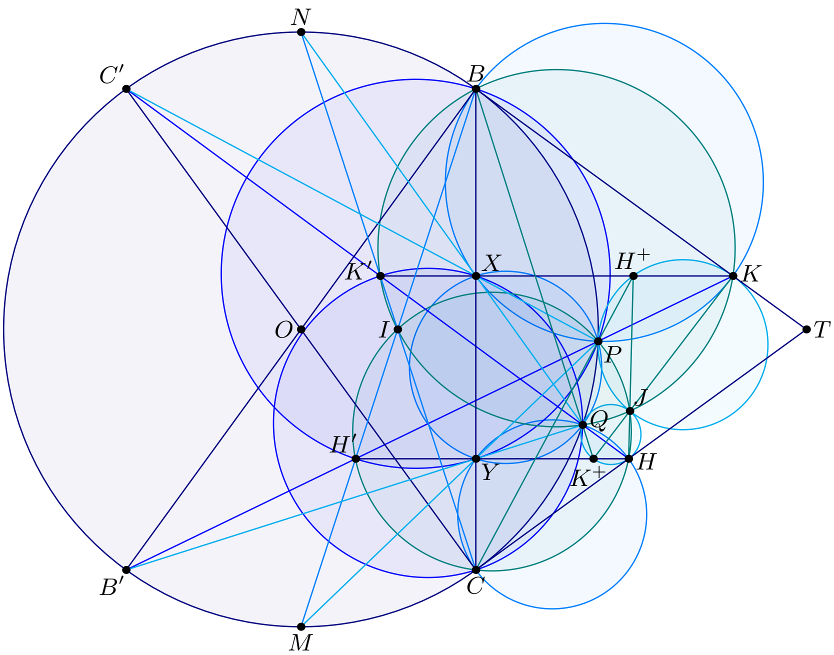
§2.29 Iran TST 2011/6

Problem 29 (Iran TST 2011/6)

Let ω be a circle with center O, and let T be a point outside of ω. Points B and C lie on ω such that $\overline{TB}$ and $\overline{TC}$ are tangent to ω. Select two points K and H on $\overline{TB}$ and $\overline{TC}$, respectively.

(a) Lines BO and CO meet ω again at B' and C', and points K' and H' lie on the angle bisectors of $\angle BCO$ and $\angle CBO$, respectively, such that $\overline{KK'}$ and $\overline{HH'}$ are perpendicular to $\overline{BC}$. Prove that K, H', B' are collinear if and only if H, K', C' are collinear.

(b) Let I be the incenter of $\triangle OBC$. Two circles in the interior of $\triangle TBC$ are externally tangent to ω and externally tangent to each other at J. Given that one of them is tangent to $\overline{TB}$ at K and the other is tangent to $\overline{TC}$ at H, prove that quadrilaterals $BKJI$ and $CHJI$ are cyclic.



First solution to part (a), by angle chasing Ignore the collinearities for now. Let $\overline{B'K}$ and $\overline{C'H}$ intersect ω again at P and Q, and let $\overline{BC}$ intersect $\overline{KK'}$ at X and $\overline{HH'}$ at Y. Denote by M and N the midpoints of arcs CB' and BC', so that $H' \in \overline{BM}$ and $K' \in \overline{CN}$.

Claim 1. Points Q, Y, B' are collinear; points P, X, C' are collinear; and $XPQY$ is cyclic.

Proof. Since $\measuredangle CQY = \measuredangle CHY = 90^\circ - \measuredangle YCH = \measuredangle OCB = \measuredangle CBB' = \measuredangle CQB'$, points Q, Y, B' are collinear. Analogously P, X, C' are collinear. Finally the concyclicity follows from the converse of Reim's theorem. To spell it out, $\measuredangle PXY = \measuredangle PC'B' = \measuredangle PQB' = \measuredangle PQY$. □

Claim 2. Points H, K', C' are collinear if and only if points Q, X, N are collinear. Symmetrically, points K, H', B' are collinear if and only if points P, Y, M are collinear.

Proof. Check that $\measuredangle K'QX = \measuredangle K'CX = \measuredangle NCB = \measuredangle C'QN$, so

$$\overline{HK'C'} \iff \measuredangle C'QX = \measuredangle K'QX \iff \measuredangle C'QX = \measuredangle C'QN \iff \overline{QXN},$$

where $\overline{UVW}$ is the assertion that U, V, W are collinear. □

Claim 3. Points Q, X, N are collinear if and only if points P, Y, M are collinear.

Proof. Assume that Q, X, N are collinear. It follows that $\measuredangle C'PY = \measuredangle XPY = \measuredangle XQY = \measuredangle NQB' = \measuredangle C'QN$, proving the claim. □

Putting these claims together yields the desired conclusion.

Second solution to part (a), by moving points Let M and N be the midpoints of arcs CB' and BC' respectively. Choose a point K on $\overline{BT}$; we construct K' on $\overline{CN}$ with $\overline{KK'} \perp \overline{BC}$, let $H = \overline{CT} \cap \overline{C'K'}$, construct H' on $\overline{BM}$ with $\overline{HH'} \perp \overline{BC}$, and let $K_0 = \overline{BT} \cap \overline{B'H'}$. Our task is to prove that $K = K_0$.

Move K along line BT at a linear rate, and let $\infty_{\perp BC}$ be the point at infinity perpendicular to $\overline{BC}$.

- By projection through $\infty_{\perp BC}$, K' moves along $\overline{CN}$ at a linear rate.
- By projection through C', H moves along $\overline{CT}$ at a linear rate.
- By projection through $\infty_{\perp BC}$, H' moves along $\overline{BM}$ at a linear rate.
- By projection through B', K_0 moves along $\overline{BT}$ at a linear rate.

Thus it suffices to verify the hypothesis for three values of K. We verify the following special cases:

- $K = B$: Then K' lies on $\overline{BC}$, to H is the reflection of C over T. Then $H' = B$, so $K_0 = B$.
- K is the reflection of B over T: Then $K' = C$, so $H = C$ and H' is a point on $\overline{CB'}$. Thus $K_0 = K$.
- K is the point at infinity along $\overline{TB}$: Then K' is the point at infinity along $\overline{CN}$, so $H = \overline{TC} \cap \overline{MC'}$. Then since H' lies on $\overline{BM}$, it is the reflection of H over $\overline{MN}$, so $\overline{B'H'}$ is tangent to ω. It follows that $K_0 = K$.

This completes the proof.

Solution to part (b) First we present a well-known lemma.

Lemma (Folklore)

Circles Γ_1 and Γ_2 are externally tangent at P, circles Γ_2 and Γ_3 are externally tangent at Q, and circles Γ_3 and Γ_1 are externally tangent at R. Let A be an arbitrary point on Γ_1. Line AP intersects Γ_2 again at B, line BQ intersects Γ_3 again at C, and line $\overline{CR}$ intersects Γ_1 again at D. Then $\overline{AD}$ is a diameter of Γ_1.

Proof. Let O_1, O_2, O_3 be the centers of Γ_1, Γ_2, Γ_3, respectively. Since D must be the insimilicenter of Γ_1 and Γ_2, rays O_1A and O_2B are parallel but in opposite directions. Similarly rays O_2B and O_3C are opposite and parallel, and so are rays O_3C and O_1D.

Combining these, $\overline{O_1A}$ and $\overline{O_1D}$ are parallel but in opposite directions; in other words, $\overline{AD}$ must be a diameter of Γ_1, as claimed. □

For now, ignore the two extra circles. We prove another concyclicity.

Claim 4. $BKQIK'$ and $CHPIH'$ are cyclic.

Proof. Since $\measuredangle BKK' = 90^\circ - \measuredangle XBK = 90^\circ - \measuredangle CC'B = \measuredangle BCC' = \measuredangle BQK'$, quadrilateral $BKQK'$ is cyclic. Furthermore $\measuredangle BIK' = \measuredangle BIN = \measuredangle BON = \measuredangle BQC' = \measuredangle BQK'$, as desired. $\square$

Claim 5. (JPK) and ω are tangent at P. Analogously (JQH) and ω are tangent at Q.

Proof. The tangent to (JPK) at K is $\overline{BT}$, which is parallel to the tangent to ω at B'. Thus by homothety, the tangency point between (JPK) and ω lies on $\overline{KB'}$ and therefore must be P. The other case follows in a similar fashion. $\square$

Let (JPK) intersect $\overline{KK'}$ again at H^+ and let (JQH) intersect $\overline{HH'}$ again at K^+.

Claim 6. J lies on $\overline{KK^+}$ and $\overline{HH^+}$.

Proof. Angle chasing, $\measuredangle KJH^+ = \measuredangle BKH^+ = 90^\circ - \measuredangle CBT$, and similarly $\measuredangle K^+JH = 90^\circ - \measuredangle TCB$. These are equal, so $\measuredangle KJH^+ = \measuredangle K^+JH$. Since $\overline{KH^+} \parallel \overline{HK^+}$, we must have that $J = \overline{KK^+} \cap \overline{HH^+}$. $\square$

By the lemma applied to (JPK), (JQH), ω, we find that the second intersection of $\overline{QK^+}$ and ω must be the antipode of B'; id est it is B. It follows that B, Q, K^+ are collinear, and by symmetry so are C, P, H^+.

To finish, remark that $\measuredangle BQJ = \measuredangle K^+QJ = \measuredangle JPK = \measuredangle BKJ$. Symmetry implies that $BKJQIK'$ and $CHJPKH'$ are cyclic, so we are done.

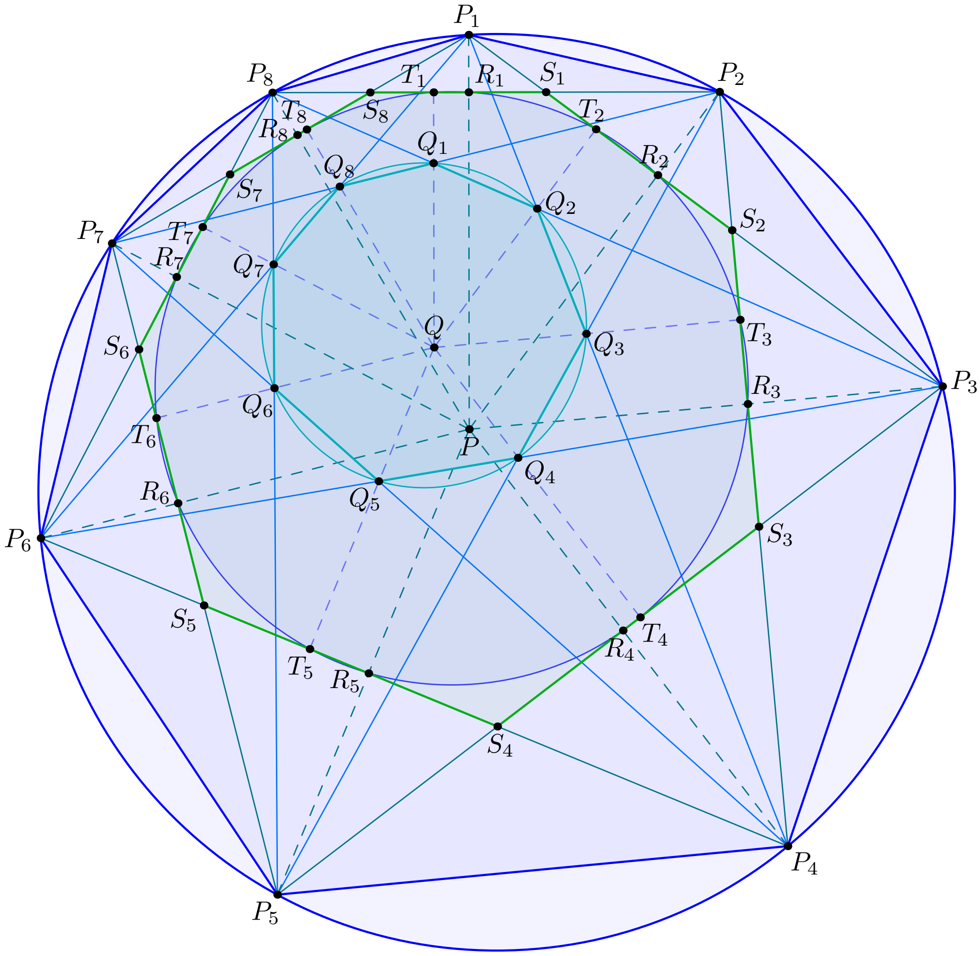
§2.30 USA TST 2020/6 (Michael Ren)

Problem 30 (USA TST 2020/6)

Let $P_1P_2\cdots P_{100}$ be a cyclic 100-gon, and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $P_{i-2}P_{i+1}$ and $P_{i-1}P_{i+2}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points $Q_1, Q_2, \ldots, Q_{100}$ are concyclic.

Replace 100 with n. Shown below is an example for $n = 8$. Let $R_i = \overline{PP_i} \cap \overline{P_{i-1}P_{i+1}}$ be the foot from P to $\overline{P_{i-1}P_{i+1}}$, and let $S_i = \overline{P_{i-1}P_{i+1}} \cap \overline{P_iP_{i+2}}$.



Claim 1. $R_1R_2\cdots R_n$ is cyclic.

Proof. Since $\angle P_iR_iP_{i+1} = \angle P_iR_{i+1}P_{i+1} = 90^\circ$ for each i, we have $PP_i \cdot PR_i = PP_{i+1} \cdot PR_{i+1}$, meaning inversion at P with radius $\sqrt{PP_i \cdot PR_i}$, which is fixed, sends the circumcircle of $P_1P_2\cdots P_n$ to that of $R_1R_2\cdots R_{100}$. □

Claim 2. $\overline{Q_iQ_{i+1}} \parallel \overline{R_iR_{i+1}}$ for all i.

Proof. By Reim's theorem on $(P_iR_iR_{i+1}P_{i+1})$ and $(P_{i-1}P_iP_{i+1}P_{i+2})$, $\overline{Q_iQ_{i+1}} = \overline{P_{i-1}P_{i+2}} \parallel \overline{R_iR_{i+1}}$. □

Since P has a pedal circle in $S_1S_2\cdots S_{100}$, the reflection Q over the center of the pedal circle is the isogonal conjugate of P in $S_1S_2\cdots S_{100}$. Let T_i be the foot from Q to $\overline{P_{i-1}P_{i+1}}$, so that there is a fixed circle through both R_i and T_i for all i.

Claim 3. $\triangle P_{i-1}P_iP_{i+1}$ and $\triangle T_{i-1}T_iT_{i+1}$ are homothetic.

Proof. By Reim's theorem on $(P_iR_iR_{i-1}P_{i-1})$ and $(R_iT_iT_{i-1}R_{i-1})$, we have $\overline{P_iP_{i-1}} \parallel \overline{T_iT_{i-1}}$, and similarly $\overline{P_iP_{i+1}} \parallel \overline{T_iT_{i+1}}$. Also $P_iS_{i-1} \cdot P_iR_{i-1} = P_iR_i \cdot P_iP = P_iS_{i+1} \cdot P_iR_{i+1}$, so $R_{i-1}S_{i-1}S_iR_{i+1}$ is cyclic, and $\overline{P_{i-1}P_{i+1}} \parallel \overline{R_{i-1}R_{i+1}}$ by Reim's theorem on $(R_{i-1}S_{i-1}S_iR_{i+1})$ and $(R_{i-1}T_{i-1}T_{i+1}R_{i+1})$. □

Claim 4. $\overline{QS_i} \perp \overline{Q_iQ_{i+1}}$ for all i.

Proof. Since S_i is the orthocenter of $\triangle PP_iP_{i+1}$, we know $\overline{PS_i} \perp \overline{P_iP_{i+1}}$. Note that $\overline{S_iP}$ and $\overline{S_iQ}$ are isogonal wrt. $\angle P_iS_iP_{i+1}$, and $\overline{P_iP_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ are isogonal, so $\overline{QS_i} \perp \overline{P_{i-1}P_{i+2}}$, as claimed. □

Claim 5. Q, Q_i, T_i are collinear for all i.

Proof. By the parallel lines from Claim 3,

$$\frac{T_iS_{i-1}}{S_{i-1}P_{i-1}} = \frac{T_iT_{i-1}}{P_iP_{i-1}} = \frac{T_iT_{i+1}}{P_iP_{i+1}} = \frac{T_iS_i}{S_iP_{i+2}}.$$

Hence a homothety at S sends $\triangle Q_iP_{i-1}P_{i+1}$ to $\triangle H_iS_{i-1}S_i$ for some point H_i with $\overline{QS_i} \perp \overline{H_iS_{i-1}}$ and $\overline{QS_{i-1}} \perp \overline{H_iS_i}$. It follows that H_i is the orthocenter of $\triangle Q_iS_{i-1}S_i$, so H_i lies on $\overline{QT_i}$. This is sufficient, since $H_i \in \overline{T_iQ_i}$ by the homothety. □

From this, $\triangle PR_iR_{i+1}$ and $\triangle QQ_iQ_{i+1}$ for each i, so $R_1R_2\cdots R_n \sim Q_1Q_2\cdots Q_n$. Since the former is cyclic, so is the latter, thus concluding the proof.