USAMO 2021

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§0 Problems

Problem 1. Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

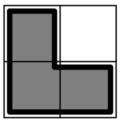
Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

Problem 2. The Planar National Park is a subset of the Euclidean plane consisting of several trails which meet at junctions. Every trail has its two endpoints at two different junctions whereas each junction is the endpoint of exactly three trails. Trails only intersect at junctions (in particular, trails only meet at endpoints). Finally, no trails begin and end at the same two junctions.

A visitor walks through the park as follows: she begins at a junction and starts walking along a trail. At the end of that first trail, she enters a junction and turns left. On the next junction she turns right, and so on, alternating left and right turns at each junction. She does this until she gets back to the junction where she started. What is the largest possible number of times she could have entered any junction during her walk, over all possible layouts of the park?

Problem 3. Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.



- If all cells in a column have a stone, you may remove all stones from that column.
- If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some nonzero number of moves, the board has no stones?

Problem 4. A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s, t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

Problem 5. Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1.
\end{aligned}$$

Problem 6. Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

§1 USAMO 2021/1 (Ankan Bhattacharya)

Problem 1 (USAMO 2021/1)

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

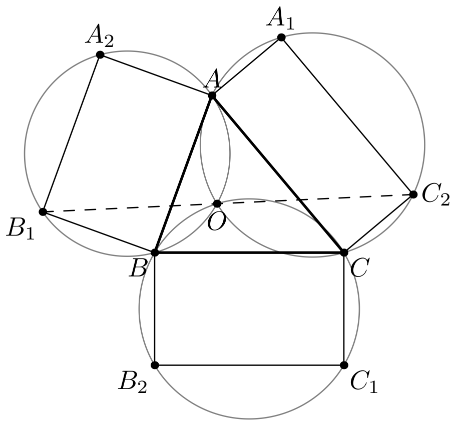
$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

Let O be the second intersection of (CAA_1C_2) and (ABB_1A_2). Observe that

$$\measuredangle BOC = \measuredangle BOA + \measuredangle AOC = \measuredangle BB_1A + \measuredangle AA_1C = \measuredangle BC_1C,$$

so O also lies on (BCC_1B_2).



Now note that $\measuredangle AOB_1 = \measuredangle AOC_2 = 90^\circ$, so O lies on $\overline{B_1C_2}$. Symmetrically, O lies on $\overline{C_1A_2}$ and $\overline{A_1B_2}$, so we are done.

§2 USAMO 2021/2

Problem 2 (USAMO 2021/2)

The Planar National Park is a subset of the Euclidean plane consisting of several trails which meet at junctions. Every trail has its two endpoints at two different junctions whereas each junction is the endpoint of exactly three trails. Trails only intersect at junctions (in particular, trails only meet at endpoints). Finally, no trails begin and end at the same two junctions.

A visitor walks through the park as follows: she begins at a junction and starts walking along a trail. At the end of that first trail, she enters a junction and turns left. On the next junction she turns right, and so on, alternating left and right turns at each junction. She does this until she gets back to the junction where she started. What is the largest possible number of times she could have entered any junction during her walk, over all possible layouts of the park?

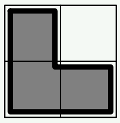
The answer is 3. The upper bound is obvious to the meanest intelligence. One construction is a pentagonal prism.

§3 USAMO 2021/3

Problem 3 (USAMO 2021/3)

Let $n \geq 2$ be an integer. An $n \times n$ board is initially empty. Each minute, you may perform one of three moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see figure; rotations not allowed), you may place a stone in each of those cells.



- If all cells in a column have a stone, you may remove all stones from that column.
- If all cells in a row have a stone, you may remove all stones from that row.

For which n is it possible that, after some nonzero number of moves, the board has no stones?

The answer is $3 \mid n$, attained by dividing the $n \times n$ board into 3×3 squares and repeating the $n = 3$ construction in all 3×3 squares simultaneously. Now we show that it is impossible to clear the board if $3 \nmid n$.

Weight the squares of the board so that the cell (i, j) has weight $x^i y^j$, as shown below:

$$\begin{bmatrix} y^{n-1} & xy^{n-1} & x^2y^{n-1} & \cdots & x^{n-1}y^{n-1} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ y^2 & xy^2 & x^2y^2 & \cdots & x^{n-1}y^2 \\ y & xy & x^2y & \cdots & x^{n-1}y \\ 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix}$$

Then adding a L-tromino is equivalent to increasing our net weight by a multiple of $1+x+y$, removing a row is equivalent to decreasing our net weight by a multiple of $1+x+\cdots+x^{n-1}$, and removing a column is equivalent to decreasing our net weight by a multiple of $1+y+\cdots+y^{n-1}$.

Thus if we eventually clear the board, there are polynomials P, Q, R, with $\deg_x P \leq n-2$ and $\deg_y P \leq n-2$, such that

$$P(x, y) \cdot (1 + x + y) = Q(x, y) \cdot (1 + x + \cdots + x^{n-1}) + R(x, y) \cdot (1 + y + \cdots + y^{n-1}).$$

In particular if $\omega_i, \omega_j \neq 1$ are nth roots of unity then

$$P(\omega_i, \omega_j) \cdot (1 + \omega_i + \omega_j) = 0.$$

For $3 \nmid n$, we never have $1 + \omega_i + \omega_j = 0$, so $P(\omega_i, \omega_j) = 0$ for all ω_i, ω_j. It should easily follow that $P(x, y) \equiv 0$, say by combinatorial nullstellensatz.

Remark (Elementary justification for $P(x, y) \equiv 0$)**.** We are given $P(\omega_i, \omega_j) = 0$ for nth roots of unity ω_i, ω_j not equal to 1.

It follows that the polynomial (in x) $P(x, \omega_j)$ has $n-1$ roots for all ω_j, but its degree is at most $n-2$, so $P(x, \omega_j) \equiv 0$. Then if $P(x, y) = A_0(y) + A_1(y) \cdot x + A_2(y) \cdot x^2 + \cdots$, we know $A_k(\omega_j) = 0$ for all ω_j, so $A_k(y) \equiv 0$, implying $P(x, y) \equiv 0$.

§4 USAMO 2021/4 (Carl Schildkraut)

Problem 4 (USAMO 2021/4)

A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

The possible values of $|S|$ are zero and powers of two.

Construction: Evidently $S = \varnothing$ works; now we construct a valid set S of 2^k elements for each $k \geq 0$.

Select arbitrary distinct primes $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k$. Then for each subset $X \subseteq \{1, \ldots, k\}$, let S contain

$$\prod_{i \in X} p_i \prod_{i \notin X} q_i.$$

It is easy to check that this works.

Proof of necessity: Observe for each $s \in S$ that $\gcd(s,t)$ is always a divisor of s; therefore, the map

$$S \to \{\text{divisors of } s\} \quad \text{by} \quad t \mapsto \gcd(s,t)$$

is a bijection.

Let p be a prime that divides some element of S, and let e be maximal so that p^e divides some element of S. I contend $e = 1$.

Assume for contradiction $e \geq 2$. Observe that exactly $\frac{1}{e+1}$ of the divisors of s are not divisible by p, so exactly $\frac{1}{e+1}$ of the elements of S are not divisible by p. However, some divisor $d \mid s$ has $\nu_p(d) = 1$, so some element $t \in S$ has $\nu_p(t) = 1$. By an analogous argument, exactly $\frac{1}{2}$ of the divisors of S are not divisible by p, implying

$$\frac{1}{e+1} = \frac{1}{2} \implies e = 1.$$

At last, each element $s \in S$ is of the form $s = \prod_{p \in P} p$ for some set of distinct primes p, so

$$|S| = |\{\text{divisors of } s\}| = 2^{|P|},$$

as desired.

§5 USAMO 2021/5

Problem 5 (USAMO 2021/5)

Let $n \ge 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1. \end{aligned}$$

The answer is

$$\begin{aligned} a_1 = a_3 = a_5 = \cdots = a_{2n-1} &= 1 \\ a_2 = a_4 = a_6 = \cdots = a_{2n} &= 2, \end{aligned}$$

which obviously works. Now we show this is the only solution.

First solution, by Cauchy-Schwarz (Ankit Bisain) First by summing the equations we obtain

$$\sum a_{\text{even}} = 2\sum a_{\text{odd}} = 4\sum \frac{1}{a_{\text{even}}}.$$

By Cauchy-Schwarz, we have

$$\left(\sum a_{\text{even}}\right)^2 = 4\left(\sum a_{\text{even}}\right)\left(\sum \frac{1}{a_{\text{even}}}\right) \ge 4n^2,$$

so $\sum a_{\text{even}} \ge 2n$ and therefore $\sum a_{\text{odd}} \ge n$.

Now for each k we have

$$a_{2k+1}^2 = \frac{a_{2k+1}}{a_{2k}} + \frac{a_{2k+1}}{a_{2k+2}} = \frac{a_{2k+1}}{a_{2k-1}+a_{2k+1}} + \frac{a_{2k+1}}{a_{2k+1}+a_{2k+3}}.$$

Summing this cyclically, we have $\sum a_{\text{odd}}^2 = n$.

Finally, by Cauchy-Schwarz again

$$n^2 = \underbrace{(1+1+\cdots+1)}_{n \text{ ones}} \cdot \left(\sum a_{\text{odd}}^2\right) \ge \left(\sum a_{\text{odd}}\right)^2 \ge n^2,$$

so equality holds and $a_1 = a_3 = \cdots = a_{2n-1}$.

It is immediate that $a_1 = a_3 = \cdots = a_{2n-1} = 1$ and $a_2 = a_4 = \cdots = a_{2n} = 2$, and we are done.

Second solution, by bounding Define the function

$$f(x) := \frac{\sqrt{x^2+8}-x}{2},$$

and consider the sequence $t_0 = 0$, $t_1 = \sqrt{2}$, $t_2 = \frac{\sqrt{10}-\sqrt{2}}{2}$, ... defined by $t_{k+1} = f(t_k)$.

Claim. For each $k \geq 0$ and each i, we have

$$t_{2k} < a_{2i+1} < t_{2k+1}.$$

Proof. The proof is by induction. Obviously $a_{2i+1} > 0 = t_0$ for all i; now we show that if $a_{2i+1} > t_k$ for all i, then $a_{2i+1} < t_{k+1}$ as well, and if $a_{2i+1} < t_k$ for all i, then $a_{2i+1} > t_{k+1}$ as well.

First suppose $a_{2i+1} > t_k$ for all i. Then

$$a_2 = a_1 + a_3 > a_1 + t_k$$
$$a_{2n} = a_1 + a_{2n-1} > a_1 + t_k,$$

thus we have

$$a_1 = \frac{1}{a_2} + \frac{1}{a_{2n}} < \frac{2}{a_1 + t_k} \implies a_1^2 + t_k \cdot a_1 - 2 < 0.$$

The quadratic formula gives $a_1 < t_{k+1}$, and applying this cyclically gives $a_{2i+1} < t_{k+1}$ for all i.

Analogously if $a_{2i+1} < t_k$ for all i, we arrive at

$$a_1^2 - t_k \cdot a_1 - 2 > 0,$$

so the quadratic formula gives $a_1 > t_{k+1}$. □

Finally, we may directly verify that for all $\varepsilon > 0$, we have

$$1 < f(1-\varepsilon) < 1 + \varepsilon/2$$
$$1 - \varepsilon/2 < f(1+\varepsilon) < 1.$$

Hence $\lim_{k\to\infty} t_{2k} = 1^-$ and $\lim_{k\to\infty} t_{2k+1} = 1^+$ (the k are integers).

This implies $a_{2i+1} = 1$ for all i, which in turn implies $a_{2i} = 2$ for all i. This completes the proof.

Third solution, by more efficient bounding (based on Andrew Gu's solution) Let $m = \min\{a_1, a_3, \ldots, a_{2n-1}\}$ and $M = \max\{a_1, a_3, \ldots, a_{2n-1}\}$. Observe from

$$a_{2k+1} = \frac{1}{a_{2k}} + \frac{1}{a_{2k+2}} = \frac{1}{a_{2k+1} + a_{2k+3}} + \frac{1}{a_{2k+1} + a_{2k-1}}$$

the bounds

$$\frac{2}{a_{2k+1} + M} \leq a_{2k+1} \leq \frac{2}{a_{2k+1} + m}.$$

It follows by selecting k with $a_{2k+1} = m$ and $a_{2k+1} = M$ that

$$M \leq \frac{2}{m+M} \leq m,$$

implying that $M = m$. Therefore $a_1 = a_3 = \cdots = a_{2n-1}$, and the conclusion readily follows.

§6 USAMO 2021/6 (Ankan Bhattacharya)

Problem 6 (USAMO 2021/6)
Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

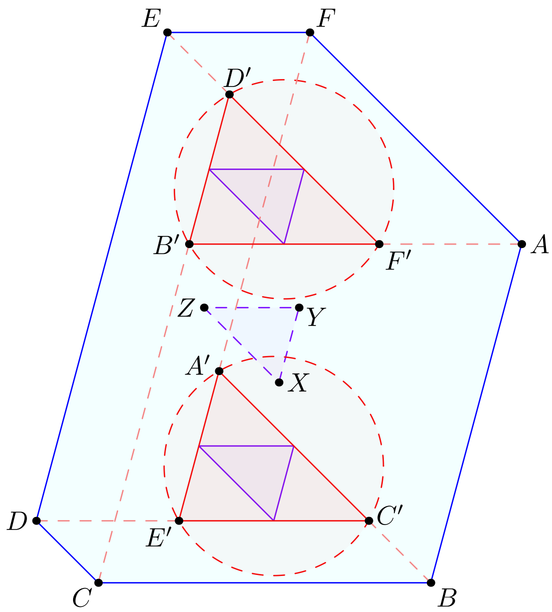
$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

In fact the orthocenter of $\triangle XYZ$ is the midpoint of the segment between the two circumcenters. We present two proofs.

First solution by parallelograms (mine) For clarity, the below diagram takes an extra degree of freedom by ignoring the length condition.

Construct parallelograms $FABA'$, $ABCB'$, $BCDC'$, $CDED'$, $DEFE'$, $EFAF'$. In general, observe that $\overline{B'F'} \parallel \overline{BC} \parallel \overline{E'C'}$ and (in directed lengths) $B'F' = CB - EF = E'C'$, so $\triangle D'B'F'$ and $\triangle A'E'C'$ are homothetic and directly congruent. This implies they are translations of each other.



Claim. If $AB \cdot DE = BC \cdot EF = CD \cdot FA$, then the circumcenters of $\triangle D'B'F'$ and $\triangle ACE$ coincide.

Proof. Observe that

$$\operatorname{Pow}(A, (D'B'F')) = AB' \cdot AF' = BC \cdot FE,$$

which is fixed, so A, C, E have equal power with respect to $(D'B'F')$. $\square$

Now let O_1 and O_2 be the circumcenters of $\triangle D'B'F'$ and $\triangle A'C'E'$. I contend that in general, the midpoint of $\overline{O_1O_2}$ coincides with the orthocenter of $\triangle XYZ$.

Let $M_DM_BM_F$ and $M_AM_EM_C$ be the medial triangles of $\triangle DBF$ and $\triangle AEC$, so their orthocenters are O_1 and O_2. It is easy to check that X, Y, Z are the midpoints of $\overline{M_DM_A}$, $\overline{M_BM_E}$, $\overline{M_FM_C}$.

But we know $\triangle M_D M_B M_F$ and $\triangle M_A M_E M_C$ are translations of each other, and $\triangle XYZ$ is their vector average, so we conclude the orthocenter of $\triangle XYZ$ is the midpoint of $\overline{O_1 O_2}$. This completes the proof.

Second solution by shared nine-point circle (author) First if points X, Y, Z coincide, then the orthocenter of $\triangle XYZ$ is the single point $X = Y = Z$, and also $\triangle ACE$ and $\triangle BDF$ are 180° rotations of each other through $X = Y = Z$, thus so are their circumcenters.

We assume henceforth X, Y, Z are not all the same point. Construct the medial triangles $N_A N_C N_E$ and $N_B N_D N_F$ of $\triangle ACE$ and $\triangle BDF$ respectively. The key claim is that the nine-point circles of $\triangle ACE$ and $\triangle BDF$ coincide; that is:

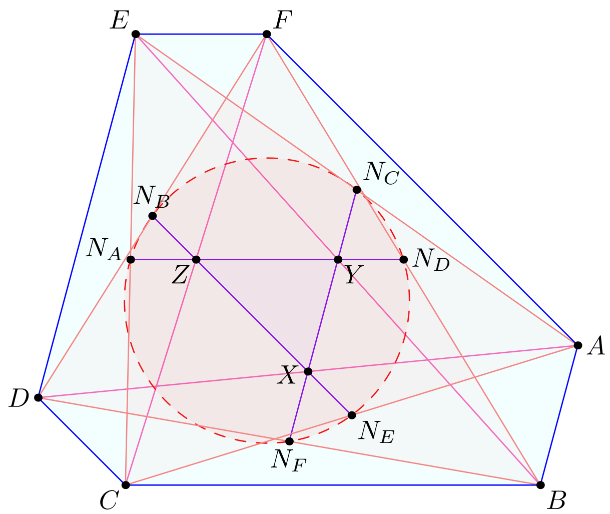
Claim. If $AB \cdot DE = BC \cdot EF = CD \cdot FA$, then the six points N_A, N_C, N_E, N_B, N_D, N_F are concyclic.

Proof. Observe that

$$XN_B \cdot XN_E = \frac{1}{2}AF \cdot \frac{1}{2}DC = \frac{1}{2}DE \cdot \frac{1}{2}AB = XN_C \cdot XN_F,$$

so N_B, N_C, N_E, N_F are concyclic.

Then if the six points are not all concyclic, by radical axis theorem on $(N_B N_C N_E N_F)$, $(N_C N_A N_F N_D)$, $(N_A N_B N_D N_E)$, the three lines $N_A N_D$, $N_B N_E$, $N_C N_F$ concur, i.e. $X = Y = Z$, contradiction. □



Now observe since

$$N_A Z = \frac{1}{2}EF = YN_D$$

that the perpendicular bisectors of $\overline{YZ}$ and $\overline{N_A N_D}$ coincide, so the circumcenter of $\triangle XYZ$ coincides with the common nine-point center.

Toss on the complex plane with this common circumcenter as the origin. Then

$$\begin{aligned}
\text{orthocenter}(\triangle XYZ) &= x + y + z = \tfrac{1}{2}(a + b + c + d + e + f) \\
\text{circumcenter}(\triangle ACE) &= \text{orthocenter}(\triangle N_A N_C N_E) = n_A + n_C + n_E = a + c + e \\
\text{circumcenter}(\triangle BDF) &= \text{orthocenter}(\triangle N_B N_D N_F) = n_B + n_D + n_F = b + d + f,
\end{aligned}$$

and so the orthocenter of $\triangle XYZ$ is the claimed midpoint.