USAMO 2019

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§0 Problems

Problem 1. Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\dots f}_{f(n)\text{ times}}(n)\dots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

Problem 2. Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side AB satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

Problem 3. Let K be the set of all positive integers that do not contain the digit 7 in their base-10 representation. Find all polynomials f with nonnegative integer coefficients such that $f(n) \in K$ whenever $n \in K$.

Problem 4. Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \dots, 2n\}$, for integers $0 \le i \le n$ and $0 \le j \le n$ such that:

- for all $0 \le i, j \le n$, the set $S_{i,j}$ has $i + j$ elements; and
- $S_{i,j} \subseteq S_{k,l}$ whenever $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

Problem 5. Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Problem 6. Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

§1 USAMO 2019/1 (Evan Chen)

Problem 1 (USAMO 2019/1)

Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\dots f}_{f(n)\text{ times}}(n)\dots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

The answer is all evens. It is not hard to check that any f that fixes the odds and is an involution on the evens works. To prove $f(1000)$ must be even, we present two solutions.

First solution, by induction In what follows, $f^k(n)$ means f iterated k times. Here we will prove all solutions to the functional equation are of the above form: f fixes odds and is an involution on the evens. Note that to prove only the original problem statement, Claims 1 and 2 suffice.

Claim 1. f is injective.

Proof. If $f(a) = f(b)$, then

$$a^2 = f^2(a) f^{f(a)}(a) = f^2(b) f^{f(b)}(b) = b^2,$$

so $a = b$ follows. $\square$

Claim 2. If n is odd, then $f(n) = n$.

Proof. We use strong induction on n, with no base case. Assume the claim holds for all odd positive integers less than n.

Consider the equation

$$f^2(n) f^{f(n)}(n) = n^2.$$

I contend both terms on the left equal n. Let $m = f^2(n)$. If $m < n$, then $f^2(n) = f^2(m)$ by inductive hypothesis, so $n = m$ by injectivity, absurd. Thus $f^2(n) \geq n$. Analogously if $m = f^{f(n)}(n)$ and $m < n$, then $f^{f(n)}(n) = f^{f(n)}(m)$, so $n = m$ by injectivity, absurd. Thus we have $f^2(n) = f^{f(n)}(n) = n$.

Now the sequence n, $f(n)$, $f^2(n)$, … repeats with period 2, so if $f(n)$ is odd, then $f(n) = f^{f(n)}(n) = n$. Otherwise suppose $f(n)$ is even, and let $m = f(n)$. Then $f(m) = n$, so $m^2 = f^2(m) f^{f(m)}(m) = n^2$, contradiction. $\square$

Claim 3. If n is even, then $f^2(n) = n$.

Proof. The proof is similar to that of Claim 2. We use strong induction on n, with no base case. Assume the claim holds for all even positive integers less than n; also recall that f fixes odds and is injective, so $f(n)$ is even for all even n.

Again consider the equation

$$f^2(n) f^{f(n)}(n) = n^2.$$

It will suffice to show, once more, both terms on the left equal n. Let $m = f^2(n)$. If $m < n$, then $f^2(n) = f^2(m)$, so $n = m$ by injectivity, absurd, so $f^2(n) \geq n$. Similarly if $m = f^{f(n)}(n)$ and $m < n$, then $f^{f(n)}(n) = m = f^{f(n)}(m)$, so $n = m$ by injectivity, absurd.

Hence $f^2(n) = n$, as needed. $\square$

Second solution, by arrows (Espen Slettnes) Just as we did above, we first show f injective. Indeed, if $f(a) = f(b)$, then $a^2 = f^2(a)f^{f(a)}(a) = f^2(b) = f^{f(b)}(b) = b^2$, so f is injective.

Consider the sequence defined by $x_0 = 1000$ and $x_i = f(x_{i-1})$ for all $i \geq 1$. Letting $n = x_i$ in the functional equation gives

$$a_{i+2}a_{i+a_{i+1}} = a_i^2. \qquad (*)$$

Take i so that a_i is minimal. Then $a_{i+2} \geq a_i$ and $a_{i+a_{i+1}} \geq a_i$, so by $(*)$ we must have $a_{i+2} = a_i$. By injectivity this reduces to $a_2 = a_0$, so $(a_i)_{i \geq 0}$ repeats with period 2.

Take $i = 0$ in $(*)$ to obtain $a_{a_1} = 1000$. Then $a_1 = 1000$ or a_1 is even. End proof.

§2 USAMO 2019/2 (Ankan Bhattacharya)

Problem 2 (USAMO 2019/2)
Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side AB satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

First solution, by symmedians There is a point P on side AB with $AP = AD^2/AB$ and $BP = BC^2/AB$. We have

$$\frac{PA}{PB} = \left(\frac{AD}{BC}\right)^2 = \left(\frac{EA}{EB}\right)^2$$

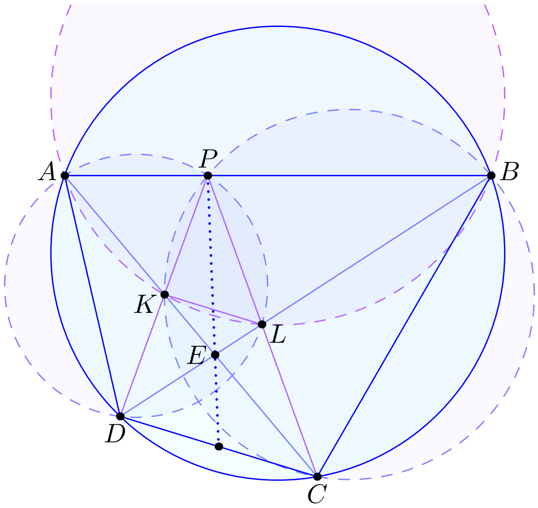
since $\triangle AED \sim \triangle BEC$. Then $\overline{EP}$ is the E-symmedian of $\triangle EAB$, so $\overline{EP}$ bisects $\overline{CD}$.

By monotonicity it remains to check $\angle APD = \angle BPC$. Since $AP \cdot AB = AD^2$, we have $\triangle APD \sim \triangle ADB$, and analogously $\triangle BPC \sim \triangle BCA$. Hence $\measuredangle APD = \measuredangle BDA = \measuredangle BCA = \measuredangle CPB$, end proof.

Second (elementary) solution, from official solutions packet As before, construct P so that $AP \cdot AB = AD^2$ and $BP \cdot BA = BC^2$. Then $\triangle APD \sim \triangle ADB$, $\triangle BPC \sim \triangle BCA$, so

$$\theta := \measuredangle APD = \measuredangle BDA = \measuredangle BCA = \measuredangle CPB.$$

The task is to show $\overline{PE}$ bisects $\overline{CD}$.



Let $K = \overline{AC} \cap \overline{PD}$, $L = \overline{BD} \cap \overline{PC}$.

Claim 1. $APLD$ and $BPKC$ are cyclic.

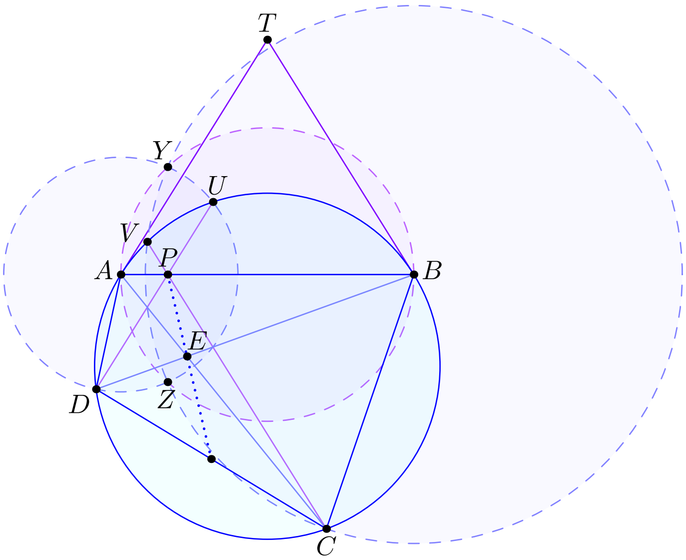
Proof. These follow from $\measuredangle LDA = \theta = \measuredangle LPA$ and $\measuredangle BCK = \theta = \measuredangle BPK$ respectively. □

Claim 2. $AKLB$ is cyclic.

Proof. We have $\measuredangle AKB = \measuredangle CKB = \measuredangle CPB = \theta$ and similarly $\measuredangle ALB = \theta$. □

By Reim's theorem on $(ABCD)$, $(AKLB)$, we have $\overline{KL} \parallel \overline{CD}$. If $M = \overline{PE} \cap \overline{CD}$, then $MC = MD$ by Ceva's theorem on $\triangle PCD$.

Third solution, by radical axes Let Ω be the circumcircle of $ABCD$, let Γ be the circle with diameter AB, and let ω_A, ω_B be the circles centered at A, B with radii AD, BC, respectively.



The condition $AD^2 + BC^2 = AB^2$ means ω_A, ω_B are orthogonal, so their intersection points Y, Z lie on Γ. Let P be the midpoint of $\overline{YZ}$. Then P lies on $\overline{AB}$, the radical axis of Ω, Γ, so P is the common radical center of Ω, Γ, ω_A, ω_B.

Let the tangents to Ω at A, B meet at T, and let $\overline{DP}$, $\overline{CP}$ meet Ω again at U, V. Since $\overline{DU}$ is the radical axis of Ω, ω_A, we have $\overline{AT} \parallel \overline{DU}$. Similarly $\overline{BT} \parallel \overline{CV}$, so $\measuredangle APD = \measuredangle BAT = \measuredangle TBA = \measuredangle CPB$. By monotonicity, P is the point described in the problem statement.

Finally we have

$$\frac{PA}{PB} = \left(\frac{YA}{YB}\right)^2 = \left(\frac{AD}{BC}\right)^2 = \left(\frac{EA}{EB}\right)^2,$$

so $\overline{EP}$ is the E-symmedian of $\triangle EAB$, which bisects $\overline{CD}$.

> **Remark.** This was my solution in-contest, and I believe it is pretty motivated. The four circles in the problem are the most natural way to construct the diagram, and an in-scale diagram suggests P lies on $\overline{YZ}$. The rest is not hard to prove.

Fourth solution, by inversion (Evan Chen) As noted above, the circle ω_A centered at A with radius AD is orthogonal to the circle ω_B centered at B with radius BC. We let $\mathbf{I}_A$, $\mathbf{I}_B$ denote inversion with respect to ω_A, ω_B.

Let the radical axis of ω_A, ω_B intersect $\overline{AB}$ at P; by design, $P = \mathbf{I}_A(B) = \mathbf{I}_B(A)$. This already implies that

$$\measuredangle APD \overset{\mathbf{I}_A}{=} \measuredangle BDA = \measuredangle BCA \overset{\mathbf{I}_B}{=} \measuredangle CPB,$$

so by monotonicity, P is the point described in the problem statement.

> **Claim.** The point $K = \mathbf{I}_A(C)$ lies on ω_B and $\overline{DP}$. Similarly $L = \mathbf{I}_B(D)$ lies on ω_A and $\overline{CP}$.

Proof. Since $\omega_A \perp \omega_B$, it follows that $K \in \omega_B$. For $K \in \overline{DP}$, note $ABCD$ is cyclic, so $P = \mathbf{I}_A(B)$, $K = \mathbf{I}_A(C)$, $D = \mathbf{I}_A(D)$ are collinear. □

Finally since C, L, P collinear, A is concyclic with $K = \mathbf{I}_A(C)$, $L = \mathbf{I}_A(L)$, $B = \mathbf{I}_A(B)$, i.e. $AKLB$ is cyclic. Hence $\overline{KL} \parallel \overline{CD}$ by Reim's theorem, and $\overline{PE}$ bisects $\overline{CD}$ by Ceva's theorem.

§3 USAMO 2019/3 (Titu Andreescu, Cosmin Pohoata, Vlad Matei)

Problem 3 (USAMO 2019/3)

Let K be the set of all positive integers that do not contain the digit 7 in their base-10 representation. Find all polynomials f with nonnegative integer coefficients such that $f(n) \in K$ whenever $n \in K$.

It is easy to verify these functions work: $f \equiv d$, $f(n) \equiv 10^e n$, and $f(n) \equiv 10^e n + d$ for $e \geq 0$, $d < 10^e$, and $d \in K$. We prove they are the only solutions.

Say a function is *happy* if the problem condition is satisfied. The key is the tackle monomials first. In particular, we claim:

Lemma (Monomial case)

If a nonconstant function of the form $f(n) \equiv cn^d$ is happy, then c is a power of 10 and $d = 1$.

The lemma follows from the following two claims:

Claim 1. If $f(n) = cn$ is happy, then c is a power of 10.

Proof. We have 1, c, c^2, c^3, ... are all elements of K, but if $\log_{10} c$ is irrational, then for some k we have $\{\log_{10} c^k\} \in [\log_{10} 7, \log_{10} 8)$; that is, the leftmost digit of c^k is 7, contradiction. □

Claim 2. If $f(n) = cn^d$ is happy, then $d \in \{0, 1\}$.

Proof. Assume $d > 0$. If $n \in K$, then $10^e n + 3 \in K$ for each $e \geq 1$; in particular,

$$K \ni f(10^e n + 3) = c \cdot 3^d + cd \cdot 3^{d-1} \cdot 10^e n + c\binom{d}{2} \cdot 3^{d-2} \cdot 10^{2e} n^2 + \cdots,$$

so by taking e sufficiently large, $cd \cdot 3^{d-1} n \in K$ for all n. Thus the function $g(n) = cd \cdot 3^{d-1} n$ is happy, so $cd \cdot 3^{d-1}$ is a power of 10 by the first claim. Then $d = 1$ follows. □

Having established the lemma, we now solve the general case. Let $f(n) = a_0 + a_1 n + a_2 n^2 + \cdots$, and for each n, plug in $f(10^e n)$ where $10^e > a_k n^k$ for each k. It follows that each of the monomials a_0, $a_1 n$, $a_2 n^2$, ... are elements of K, so the function $g_k(n) \equiv a_k n^k$ is happy for each k.

It follows that either $f \equiv d$ or $f(n) \equiv 10^e n + d$ for some e, d, and we can directly check that if $d \neq 0$, then $d < 10^e$ and $d \in K$.

§4 USAMO 2019/4 (Ricky Liu)

Problem 4 (USAMO 2019/4)

Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, for integers $0 \le i \le n$ and $0 \le j \le n$ such that:

- for all $0 \le i, j \le n$, the set $S_{i,j}$ has $i + j$ elements; and
- $S_{i,j} \subseteq S_{k,l}$ whenever $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

The answer is $(2n)! \cdot 2^{n^2}$.

First solution, by direct computation Place the sets in an $(n+1) \times (n+1)$ grid shown below, and select the blue sets $S_{0,0}$, $S_{0,1}$, $\ldots$, $S_{0,n}$, $S_{1,n}$, $\ldots$, $S_{n,n}$. Each admits exactly one more element than the previous set in the sequence, so there are $(2n)!$ ways to select the blue sets.

$$\begin{bmatrix} 1357 & 12357 & 123457 & 1234578 & 12345678 \\ 135 & 1345 & 13457 & 134578 & 1345678 \\ 13 & 135 & 1357 & 13578 & 135678 \\ 1 & 15 & 157 & 1578 & 15678 \\ \varnothing & 5 & 57 & 578 & 5678 \end{bmatrix}$$

It remains to select the remaining sets. First note that the conditions given in the problem statement are equivalent to $S_{i,j} \subset S_{i+1,j}$ and $S_{i,j} \subset S_{i,j+1}$ for all $0 \le i, j \le n$ (by transitivity).

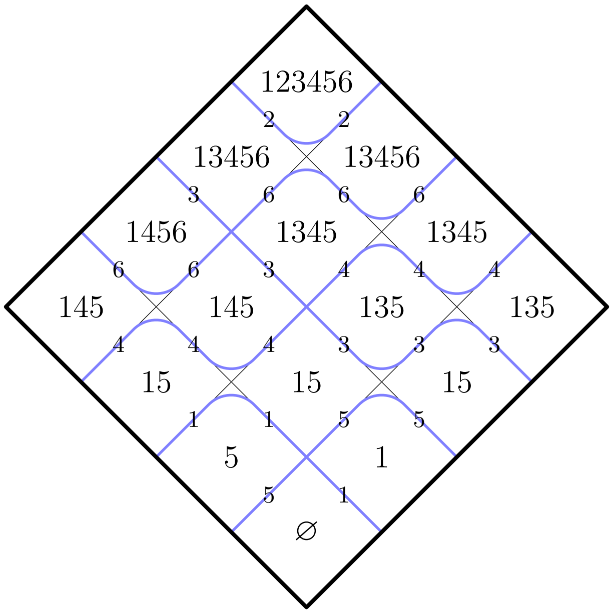
We determine the remaining sets in the following order: $S_{1,n-1}$, $S_{1,n-2}$, $\ldots$, $S_{1,0}$, $S_{2,n-1}$, $S_{2,n-2}$, $\ldots$, $S_{2,0}$, and so on. When it comes to determine $S_{i,j}$, we will have already chosen $S_{i-1,j}$ and $S_{i,j+1}$.

$$\begin{bmatrix} 1357 & 12357 \\ 135 & 1345 \end{bmatrix}$$

It is given $S_{i,j} \setminus S_{i-1,j}$ is a single-element subset of $S_{i,j+1} \setminus S_{i-1,j}$, which has cardinality 2; hence we have 2 choices.

There were $(2n)!$ ways to select the initially blue sets, and 2^{n^2} ways to select the remaining sets. The conclusion follows.

Second solution, by bijection (Daniel Zhu) Arrange the sets into the obvious $(n+1) \times (n+1)$ square grid. Here, we orient the grid diagonally so that $S_{0,0}$ is on the bottom and $S_{n,n}$ is on the top. Label an internal edge with i if crossing an edge while going up adds the element i to the set in the relevant square.



It follows from the $S_{i,j} \subseteq S_{k,\ell}$ condition that the set of edges with label i must follow a path from the left end of the square to the right end of the square, so we can associate every valid arrangement of sets with a partitioning of the internal edges into $2n$ paths, labeled 1, 2, $\ldots$, $2n$, traveling from the left end of the square to the right end of the square. Indeed, this correspondence is reversible, since given the system of labeled paths one can associate a square with the set of all numbers k so that the path labeled with k passes below the square.

It suffices to show that there are 2^{n^2} ways to choose all the ways to partition the unlabeled edges into $2n$ unlabeled paths. However, since every edge is utilized, this is equivalent to, for each of the n^2 internal vertices, choosing whether the two paths that pass through the vertex end up crossing or not. Thus, we are done.

§5 USAMO 2019/5 (Yannick Yao)

Problem 5 (USAMO 2019/5)

Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

This is possible if and only if $m+n$ is a power of 2. In what follows, let $r = \frac{m}{n}$, so we begin with r and $\frac{1}{r}$ on the board.

Proof of sufficiency: Assume $m+n$ is a power of 2. It is possible to write 1 on the board only by taking arithmetic means.

Claim 1. If x and y are on the board, then for all nonnegative integers u, v, t with $u+v=2^t$, it is possible to construct

$$\frac{u}{2^t} \cdot x + \frac{v}{2^t} \cdot y.$$

Proof. The proof is by induction on t, with $t = 0$ given. Now suppose the hypothesis holds for t; we will show it holds for $t+1$.

For $u+v = 2^{t+1}$, if $0 \in \{u, v\}$, then we are already done. Otherwise, note that

$$\frac{u}{2^{t+1}} \cdot x + \frac{v}{2^{t+1}} \cdot y = \frac{1}{2}\left[\left(\frac{1}{2^t} \cdot x + \frac{1}{2^t} \cdot y\right) + \left(\frac{u-1}{2^t} \cdot x + \frac{v-1}{2^t} \cdot y\right)\right],$$

which is constructable. □

Finally take $x = r$, $y = \frac{1}{r}$, $u = n$, $v = m$. We can construct

$$\frac{n}{m+n} \cdot r + \frac{m}{m+n} \cdot \frac{1}{r} = 1.$$

Proof of necessity: Suppose p is an odd prime dividing $m+n$. Then both numbers on the board are equivalent to $-1 \pmod p$. The main idea is in the following claim:

Claim 2. If $x \equiv y \equiv -1 \pmod p$, then both their arithmetic mean and harmonic mean are $-1 \pmod p$.

Proof. This is fairly obvious: since $2 \not\equiv 0 \pmod p$ and $x + y \equiv -2 \not\equiv 0 \pmod p$, we can check

$$\frac{x+y}{2} \equiv \frac{-2}{2} \equiv -1 \pmod p \quad \text{and} \quad \frac{2xy}{x+y} \equiv \frac{2}{\frac{1}{x}+\frac{1}{y}} \equiv \frac{2}{-2} \equiv -1 \pmod p,$$

and the claim follows. □

If Evan can write 1 on the board, then $1 \equiv -1 \pmod p$, which is absurd.

§6 USAMO 2019/6 (Titu Andreescu, Gabriel Dospinescu)

Problem 6 (USAMO 2019/6)

Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

The answer is $P(x) \equiv A(x^2 + 3)$ for any real number A, which all work. We now prove these are the only solutions.

Consider the polynomial

$$\begin{aligned} Q(x, y, z) &:= xP(x) + yP(y) + zP(z) \\ &\quad - xyz\big[P(x-y) + P(y-z) + P(z-x)\big]. \end{aligned}$$

The main idea is this:

Claim. $Q(x, y, z) = 0$ for all complex numbers x, y, z satisfying $2xyz + x + y + z$.

Proof. We are given $Q(x, y, z) = 0$ for all real numbers x, y, z satisfying $2xyz = x + y + z$, thus the rational function

$$R(x, y) := Q\left(x, y, \frac{x+y}{2xy-1}\right)$$

has infinitely many roots, i.e. $R \equiv 0$. □

Since $Q(x, -x, 0) = 0$, we have P is even. Also note

$$\begin{aligned} 0 &= Q\left(x, \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right) \\ \implies xP(x) &= \frac{x}{2}\left[P\left(x + \frac{i}{\sqrt{2}}\right) + P\left(x - \frac{i}{\sqrt{2}}\right) + P(\sqrt{2}i)\right] \\ \implies -P(\sqrt{2}i) &\equiv P\left(x + \frac{i}{\sqrt{2}}\right) + P\left(x - \frac{i}{\sqrt{2}}\right) - 2P(x). \end{aligned}$$

The right-hand expression is a second-order finite difference, so it has degree $\deg P - 2$. The left-hand expression is constant, so $\deg P \in \{0, 2\}$, and $P(x) \equiv Ax^2 + B$ for some A, B.

Finally check that

$$2A - B = -P(\sqrt{2}i) = A\left[\left(x + \frac{i}{\sqrt{2}}\right)^2 + \left(x - \frac{i}{\sqrt{2}}\right)^2 - 2x^2\right] = -A,$$

so $A = 3B$, as needed.