USAMO 2018

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Last updated June 12, 2020

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§0 Problems

Problem 1. Let a, b, c be positive real numbers such that $a+b+c = 4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

Problem 2. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all positive real numbers x, y, z with $xyz = 1$.

Problem 3. For a given integer $n \geq 2$, let $\{a_1, a_2, \ldots, a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by m for every positive integer k.

Problem 4. Let p be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k,\ a_2 + 2k,\ \ldots,\ a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

Problem 5. In convex cyclic quadrilateral $ABCD$, lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90^\circ$.

Problem 6. Let a_n be the number of permutations $(x_1, x_2, \ldots, x_n)$ of the numbers $(1, 2, \ldots, n)$ such that the n ratios $\frac{x_k}{k}$ for $1 \leq k \leq n$ are all distinct. Prove that a_n is odd for all $n \geq 1$.

§1 USAMO 2018/1 (Titu Andreescu)

Problem 1 (USAMO 2018/1)

Let a, b, c be positive real numbers such that $a+b+c = 4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca) + 4\min(a^2, b^2, c^2) \geq a^2+b^2+c^2.$$

Without loss of generality $a \leq b \leq c$, and by homogeneity let $abc = 1$, i.e. $a+b+c = 4$. Then observe the following, where the first inequality follows from AM-GM:

$$\begin{aligned} 4a + \tfrac{1}{a} &\geq 4 \\ \implies a(a+b+c) + bc &\geq 4 \\ \implies 4(a^2+ab+bc+ca) &\geq (a+b+c)^2 \\ \implies 2(ab+bc+ca) + 4a^2 &\geq a^2+b^2+c^2, \end{aligned}$$

Remark. Equality holds when $a = 1/2$, $bc = 2$, $b+c = 7/2$, i.e.

$$(a:b:c) = \left(\frac{1}{2} : \frac{7-\sqrt{17}}{4} : \frac{7+\sqrt{17}}{4}\right)$$

and permutations.

§2 USAMO 2018/2 (Nikolai Nikolov, Titu Andreescu)

Problem 2 (USAMO 2018/2)

Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f\left(x+\frac{1}{y}\right)+f\left(y+\frac{1}{z}\right)+f\left(z+\frac{1}{x}\right)=1$$

for all positive real numbers x, y, z with $xyz = 1$.

Consider the substitutions

$$g(x) := f\left(\frac{1}{x}-1\right) \quad \text{and} \quad h(x) := g\left(x+\frac{1}{3}\right)-\frac{1}{3}.$$

We have the following properties:

$$\begin{aligned} g : (0,1) \to (0,1). \qquad & g(x)+g(y)+g(z)=1 \quad \text{for} \quad x+y+z=1. \\ h : (-\tfrac{1}{3},\tfrac{2}{3}) \to (-\tfrac{1}{3},\tfrac{2}{3}). \qquad & h(x)+h(y)+h(z)=0 \quad \text{for} \quad x+y+z=0. \end{aligned}$$

I contend all valid h obey $h(x) \equiv kx$ for some constant k. Such h obviously work when $k \in [\frac{1}{2}, 1]$ (and no other k work), so we verify h is of this form. We proceed in four steps:

- By $z = 0$, we find h is odd.
- From above, we have $h(x+y) = -h(-x) - h(-y) = h(x) + h(y)$ for $x, y \in (-\frac{1}{6}, \frac{1}{6})$, so by bounded Cauchy, for some k we have $h(x) = kx$ for x on that interval.
- For $x \in (-\frac{1}{3}, \frac{1}{3})$, note $h(x) = -2h(-\frac{1}{2}x) = 2h(\frac{1}{2}x) = kx$.
- For $x \in [\frac{1}{3}, \frac{2}{3})$, again note $h(x) = -2h(-\frac{1}{2}x) = 2h(\frac{1}{2}x) = kx$, as needed.

Unravelling the substitutions, the answer is

$$f(x) \equiv \frac{k}{x+1}+\frac{1-k}{3},$$

for each constant $k \in [\frac{1}{2}, 1]$.

§3 USAMO 2018/3 (Ivan Borensco)

Problem 3 (USAMO 2018/3)

For a given integer $n \geq 2$, let $\{a_1, a_2, \ldots, a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by m for every positive integer k.

For convenience, let $A(n)$ be the set of positive integers less than n that are relatively prime to n, so that $|A(n)| = \varphi(n)$.

We begin with a lemma:

Lemma

For each n, j, and prime p, we have

$$\nu_p\left(\sum_{x=1}^{n} x^j\right) \geq \nu_p(n) - 1.$$

Proof. Let $e = \nu_p(n)$. Note the following:

$$\sum_{x=1}^{n} x^j \equiv \frac{n}{p^e} \sum_{x=0}^{p^e-1} x^j \pmod{p^e},$$

It will suffice to check $\nu_p\left(\sum_{x=0}^{p^e-1} x^j\right) \geq e-1$. Define

$$T_k := \sum_{\nu_p(x)=k} x^j, \quad \text{so that} \quad \sum_{x=0}^{p^e-1} x^j \equiv \sum_{k=0}^{e} T_k \pmod{p^e}.$$

I contend $\nu_p(T_k) \geq e-1$ for each k, which will suffice.

Fix k, and let g be a primitive root modulo p^{e-k}. We have

$$T_k \equiv p^{kj} \sum_{i=0}^{\varphi(p^{e-k})-1} g^{ij} \equiv \frac{g^{\varphi(p^{e-k})j} - 1}{g^j - 1} p^{kj} \pmod{p^e}.$$

There are two cases:

- If $p-1 \mid j$, then by LTE,

$$\nu_p\left(\frac{g^{\varphi(p^{e-k})j} - 1}{g^j - 1}\right) = \nu_p\left(\varphi\left(p^{e-k}\right)\right) = e-k-1,$$

  so $\nu_p(T_k) = (e-k-1) + kj \geq e-1$.

- Otherwise $g^j - 1$ contributes no powers of p, and

$$\nu_p\left(g^{\varphi(p^{e-k})j} - 1\right) \geq e-k$$

  since $g^{\varphi(p^{e-k})j} \equiv 1 \pmod{p^{e-k}}$, so $\nu_p(T_k) = (e-k) + kj \geq e$.

Thus the lemma is proven. □

Now we induct on the number of (not necessarily distinct) prime factors of n. The base case $n = 2$ is easy to check. Recall that n is always even, so we now proceed with the inductive step $n \to nq$, where q is prime.

Claim 1. If $q \mid n$ and the problem statement holds for n, then it holds for nq.

Proof. Observe that

$$A(nq) = \bigcup_{i=0}^{q-1} (A(n) + in).$$

It follows that

$$\begin{aligned}\sum_{a \in A(nq)} a^k &= \sum_{i=0}^{q-1} \sum_{a \in A(n)+in} a^k = \sum_{i=0}^{q-1} \sum_{a \in A(n)} (a+in)^k \\ &= \sum_{j=1}^{k} \left[\binom{k}{j} n^j \left(\sum_{i=0}^{q-1} i^j \right) \left(\sum_{a \in A(n)} a^{k-j} \right) \right] + q \sum_{a \in A(n)} a^k\end{aligned}$$

Recall that $\varphi(nq) = q\varphi(n)$. This is divisible by $\varphi(n)$ by the inductive hypothesis, and there are evidently enough factors of q. □

Claim 2. If $q \nmid n$ and the problem statement holds for n, then it holds for nq.

Proof. Instead, observe that

$$A(nq) = \bigcup_{i=0}^{q-1} (A(n) + in) \setminus qA(n).$$

As computed above, we have

$$\sum_{a \in A(nq)} a^k = \sum_{j=1}^{k} \left[\binom{k}{j} n^j \left(\sum_{i=0}^{q-1} i^j \right) \left(\sum_{a \in A(n)} a^{k-j} \right) \right] - \left(q^k - q\right) \sum_{a \in A(n)} a^k$$

Here $\varphi(nq) = (q-1)\varphi(n)$. As in the first claim, this is divisible by $\varphi(n)$, so it suffices to verify there are enough factors of p for $p \mid q-1$.

The term $(q^k - q) \sum_{a \in A(n)} a^k$ is divisible by $(q-1)\varphi(n)$, so it remains to check each term of the left summation. Indeed, the n^j term contributes at least 1 factor of p, the $\sum_{i=0}^{q-1} i^j$ term contributes at least $\nu_p(q-1) - 1$ factors of p (by the lemma), and the $\sum_{a \in A(n)} a^{k-j}$ contributes $\nu_p(\varphi(n))$ factors of p.

The inductive step follows. □

§4 USAMO 2018/4 (Ankan Bhattacharya)

Problem 4 (USAMO 2018/4)

Let p be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k,\ a_2 + 2k,\ \ldots,\ a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

For any two $i < j$, we have

$$a_i + ik \equiv a_j + jk \pmod{p} \iff k \equiv (a_i - a_j)(j-i)^{-1} \pmod{p}.$$

Hence, the number of (i, j, k) with $i < j$ and $a_i + ik \equiv a_j + jk \pmod{p}$ is precisely $\binom{p}{2}$.

By Pigeonhole, for some k there are at most $\frac{p-1}{2}$ pairs $i < j$ with $a_i + ik \equiv a_j + jk \pmod{p}$, thus there are at least $\frac{p+1}{2}$ distinct residues among $a_1 + k$, …, $a_p + pk$.

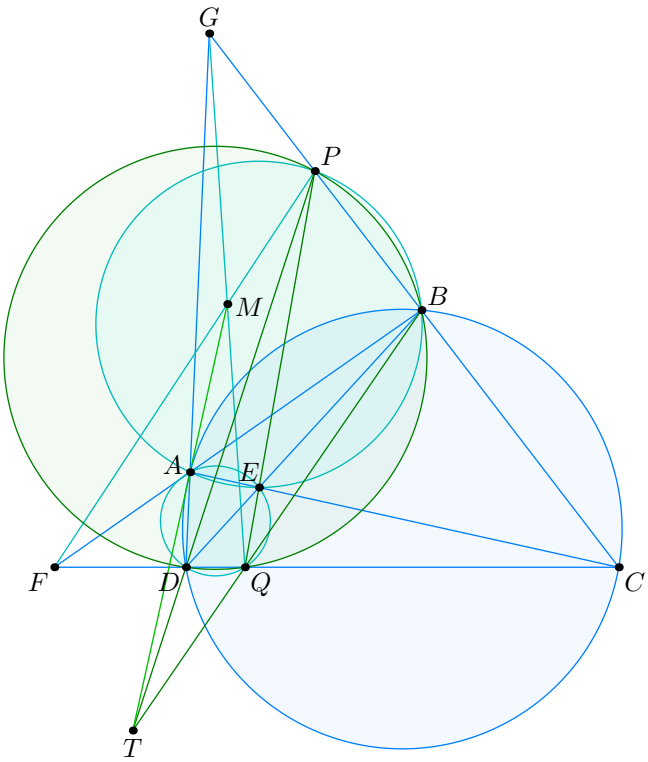
§5 USAMO 2018/5 (Kada Williams)

> **Problem 5** (USAMO 2018/5)
>
> In convex cyclic quadrilateral $ABCD$, lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90^\circ$.

First solution, by Pappus' Theorem This is a Miquel point problem with respect to quadrilateral $BPDQ$, as evidenced by the following three observations:

- $CB \cdot CP = CA \cdot CE = CD \cdot CQ$, so $BPDQ$ is cyclic.
- $\measuredangle AEP = \measuredangle ABP = \measuredangle ABC = \measuredangle ADC = \measuredangle ADQ = \measuredangle AEQ$, so E lies on $\overline{PQ}$.
- It follows that A is the Miquel point of $BDQP$.

Let $T = \overline{BQ} \cap \overline{DP}$; by properties of the Miquel point $\angle TAC = 90^\circ$. By Pappus theorem on $BQGDPF$, we have M, A, T collinear, so $\angle MAC = 90^\circ$ as well.



Second solution, by harmonic bundles First since

$$\measuredangle BAC = \measuredangle BDC = \measuredangle EDQ = \measuredangle EAQ = \measuredangle CAQ,$$

$\overline{AC}$ bisects $\angle BAQ$ and similarly $\angle DAP$. Let their common external angle bisectors intersect $\overline{BC}$ and $\overline{DC}$ at X and Y respectively. Since $-1 = (CX; PG) = (CY; FQ)$, lines XY, FP, GQ concur at M. It is clear that $\overline{AM} \perp \overline{AC}$.

§6 USAMO 2018/6 (Richard Stong)

Problem 6 (USAMO 2018/6)

Let a_n be the number of permutations $(x_1, x_2, \ldots, x_n)$ of the numbers $(1, 2, \ldots, n)$ such that the n ratios $\frac{x_k}{k}$ for $1 \le k \le n$ are all distinct. Prove that a_n is odd for all $n \ge 1$.

Say a permutation is *good* if $\frac{x_k}{k}$ are all distinct, and *bad* otherwise. The *inverse permutation* $(y_1, \ldots, y_n)$ of $(x_1, \ldots, x_n)$ is such that $y_{x_k} = k$ for all k.

Lemma 1 (Reduction to involutions)

A permutation is good if and only if its inverse permutation is good.

Proof. The proof is direct: if $(x_1, \ldots, x_n)$ is good and has inverse $(y_1, \ldots, y_n)$, then for each i, j,

$$\frac{y_i}{i} = \frac{y_i}{x_{y_i}} \neq \frac{y_j}{x_{y_j}} = \frac{y_j}{j},$$

so $(y_1, \ldots, y_n)$ is good. □

It suffices to show there is an odd number of good permutations equal to its own inverse — i.e. involutions.

If an involution is good, then it has at most one fixed point. We consider these involutions as maximal matchings on a graph of n vertices labeled 1, …, n. Label each edge $i \sim j$, where $i < j$, with the ratio i/j.

Lemma 2 (Total maximal matchings)

The number of maximal matchings of a graph with n distinguishable vertices is always odd.

Proof. Let the number of maximal matchings be $f(n)$. I claim $f(n) = (2\lceil n/2 \rceil - 1)!!$.

Indeed, $f(2k) = (2k-1)f(2k-2)$ by selecting an arbitrary vertex and its neighbor, and $f(2k+1) = (2k+1)f(2k)$ by selecting its fixed point. □

Lemma 3 (Main step)

There is an even number of bad maximal matchings.

Proof. Consider an undirected, nonsimple graph $\mathcal{G}$ on all bad maximal matchings. The key is to consider the following adjacency:

> For a matching $\mathbf{x}$ in $\mathcal{G}$, select some (possibly zero) number of disjoint pairs of edges $a \sim b$, $c \sim d$ in $\mathbf{x}$ with the same label, and swap b, c. The result is another bad maximal matching — draw an edge between it and $\mathbf{x}$.

It is easy to verify the operation above is symmetric. I contend each vertex has even degree.

Let $\mathbf{x}$ be a vertex in $\mathcal{G}$. For each λ, let m_λ denote the number of edges $a \sim b$ in $\mathbf{x}$ with $a/b = \lambda$, and let s_λ be the number of ways to swap these m_λ edges. By similar computation to Lemma 2,

$$s_\lambda = \sum_{k \ge 0} \binom{m_\lambda}{2k} (2k-1)!! \equiv \sum_{k \ge 0} \binom{m_\lambda}{2k} = 2^{m_\lambda - 1} \pmod 2,$$

which is even whenever $n_\lambda \geq 2$.

Finally $\deg \mathbf{x}$ is the product of s_λ over all $\lambda < 1$. Since $\mathbf{x}$ is bad, $n_\lambda \geq 2$ for some λ, so $\deg \mathbf{x}$ is even. Removing all self-loops, $\mathcal{G}$ is simple and each vertex of $\mathcal{G}$ now has odd degree, so $\mathcal{G}$ has an even number of vertices. □

In conclusion, the total number of maximal matchings is even, but the number of bad maximal matchings is even, so the number of good maximal matchings is odd, the end.