USAMO 2017

Compiled by Eric Shen

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Contents

§0 Problems

Problem 1. Prove that there exist infinitely many pairs of relatively prime positive integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

Problem 2. Let m_1, m_2, ..., m_n be n (not necessarily distinct) positive integers. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = (w_1, \ldots, w_n)$ of $(m_1, \ldots, m_n)$, define an A-*inversion* of ω to be a pair of indices i, j with $i < j$ for which one of the following conditions holds:

- $a_i \geq w_i > w_j$,
- $w_j > a_i \geq w_i$, or
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $(m_1, \ldots, m_n)$ having exactly k A-inversions is equal to the number of permutations of $(m_1, \ldots, m_n)$ having exactly k B-inversions.

Problem 3. Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Problem 4. Let P_1, P_2, ..., P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, none of which is $(1, 0)$. Each point is colored either red of blue, with exactly n red points and n blue points. Let R_1, R_2, ..., R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 traveling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points B_1, ..., B_n.

Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering R_1, ..., R_n of the red points.

Problem 5. Find all real numbers $c > 0$ such that there exists a labeling of the lattice points $(x, y) \in \mathbb{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and
- for each label i, the distance between any two points labeled i is at least c^i.

Problem 6. Find the minimum possible value of

$$\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4},$$

given that a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$.

§1 USAMO 2017/1 (Gregory Galperin)

Problem 1 (USAMO 2017/1)

Prove that there exist infinitely many pairs of relatively prime positive integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

For every k, the ordered pair $(2k-1, 2k+1)$ works. Both elements are relatively prime. The condition $a + b \mid a^b + b^a$ may be checked via binomial theorem as follows:

$$\begin{aligned} (2k-1)^{2k+1} &\equiv -1 + 2k(2k+1) \pmod{4k}; \\ (2k+1)^{2k-1} &\equiv -1 + 2k(2k-1) \pmod{4k}. \end{aligned}$$

Thus $(2k-1)^{2k+1} + (2k+1)^{2k-1} \equiv 0 \pmod{4k}$.

§2 USAMO 2017/2 (Maria Monks)

Problem 2 (USAMO 2017/2)

Let m_1, m_2, …, m_n be n (not necessarily distinct) positive integers. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = (w_1, \ldots, w_n)$ of $(m_1, \ldots, m_n)$, define an A-*inversion* of ω to be a pair of indices i, j with $i < j$ for which one of the following conditions holds:

- $a_i \geq w_i > w_j$,
- $w_j > a_i \geq w_i$, or
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $(m_1, \ldots, m_n)$ having exactly k A-inversions is equal to the number of permutations of $(m_1, \ldots, m_n)$ having exactly k B-inversions.

We will prove for any A that the number of permutations having exactly k A-inversions equals the number of permutations having exactly k inversions, where an inversion is a pair of indices i, j with $i < j$ and $a_i > a_j$.

The proof proceeds in two main steps:

(i) Prove the problem for $m_1 < \cdots < m_n$ (all distinct).

(ii) Extend to all $m_1 \leq \cdots \leq m_n$ (not necessarily distinct).

First, combinatorial proof of (i) Let $\mathrm{Inv}_A(i)$ be the number of $j > i$ such that (w_i, w_j) is an A-inversion. We will construct a bijection between $(w_1, \ldots, w_n)$ with k A-inversions and $(p_1, \ldots, p_n)$ with k inversions. The process is as follows:

> For each $i = n, \ldots, 1$, write the variable p_i on the board such that p_i is the $(\mathrm{Inv}_A(i)+1)$th element from the left. (For instance, we first write p_n on the board; then, if $\mathrm{Inv}_A(n-1) = 0$, we write p_{n-1} to the left of p_n, and if $\mathrm{Inv}_A(n-1) = 1$, we write p_{n-1} to the right of p_n.) Then set the ith variable from the left equal to m_i. (For instance, if the board reads p_3, p_1, p_2, then we have $p_3 = m_1$, $p_1 = m_2$, $p_2 = m_3$.)

By design, for each i, the number of $j < i$ with $p_i < p_j$ equals $\mathrm{Inv}_A(i)$. Hence the number of inversions of $(p_1, \ldots, p_n)$ equals the number of A-inversions of $(w_1, \ldots, w_n)$.

This operation is clearly injective, thus it is a bijection, and the proof is complete.

Second, inductive proof of (i), with generating functions We claim via induction on n that the generating function for the number of permutations having k A-inversions is always

$$n!_x = 1 \cdot (1+x) \cdot \left(1+x+x^2\right) \cdots \left(1+x+\cdots+x^{n-1}\right).$$

(Here, the number of permutations having k A-inversions is the coefficient of x^k.)

The base case $n = 1$ is clear. Assume the claim is true for $n-1$, and let $m_k \leq a_1 < m_{k+1}$ (with $m_0 = -\infty$, $m_{n+1} = \infty$).

- For $i \geq 0$, if $w_1 = m_{k-i}$, then there are $n-i-1$ inversions with w_1. (Namely (w_1, w_j) with $j < k-i$ or $j \geq k+1$.) Thus this case contributes a $x^{n-i-1}(n-1)!_x$ term.

- For $i > 0$, if $w_i = m_{k+i}$, then there are $i - 1$ inversions with w_1. (Namely (w_1, w_j), with $k + 1 \le j < k + i$.) Thus this case contributes a $x^i(n-1)!_x$ term.

The new generating function is then

$$n!_x = (n-1)!_x \left(\sum_{i=0}^{k-1} x^{n-i-1} + \sum_{i=1}^{n-k} x^{i-1} \right) = (n-1)!_x \left(1 + x + \cdots + x_{n-1}\right),$$

as needed.

Proof of (ii), with generating functions What follows is really a combinatorial argument, but expressing it without generating functions is a huge pain.

We reuse notation from the generating functions proof of (i), where $n!_x$ is the generating function for the number of permutations of $(m_1, \dots, m_n)$ having k A-inversions. (It is not necessary to know the explicit form for $n!_q$ that we used above.)

Let the multiset $\{m_1, \dots, m_n\}$ contain k distinct integers $\lambda_1 < \cdots < \lambda_k$ with multiplicity c_1, …, c_k respectively. Let $F(x)$ be the generating function for the number of permutations of $(m_1, \dots, m_n)$ having k A-inversions, with multiplicity (so $F(1) = n!$).

Claim. The explicit form for $F(x)$ is

$$F(x) = n!_x \cdot \frac{c_1! c_2! \cdots c_k!}{c_1!_x c_2!_x \cdots c_k!_x}.$$

Proof. Slightly perturb each m_i, replacing m_i with $m_i + i\varepsilon$, where $0 < \varepsilon \ll 1/n$. Then the generating function is $n!_x$. (Obviously both proofs to (i) still hold when $(m_1, \dots, m_n)$ are real numbers.)

Next we undo each perturbation. For $i = 1, \dots, k$, I claim the excessive ordering of the c_i instances of λ_i contribute an overcount of $c_i!_x / c_i!$. Indeed, there is a factor of $c_i!_x$ that should instead be $c_i! \cdot x^0$, since there are no inversions between equivalent elements of the permutation.

Undoing the overcounts, the claim then follows. □

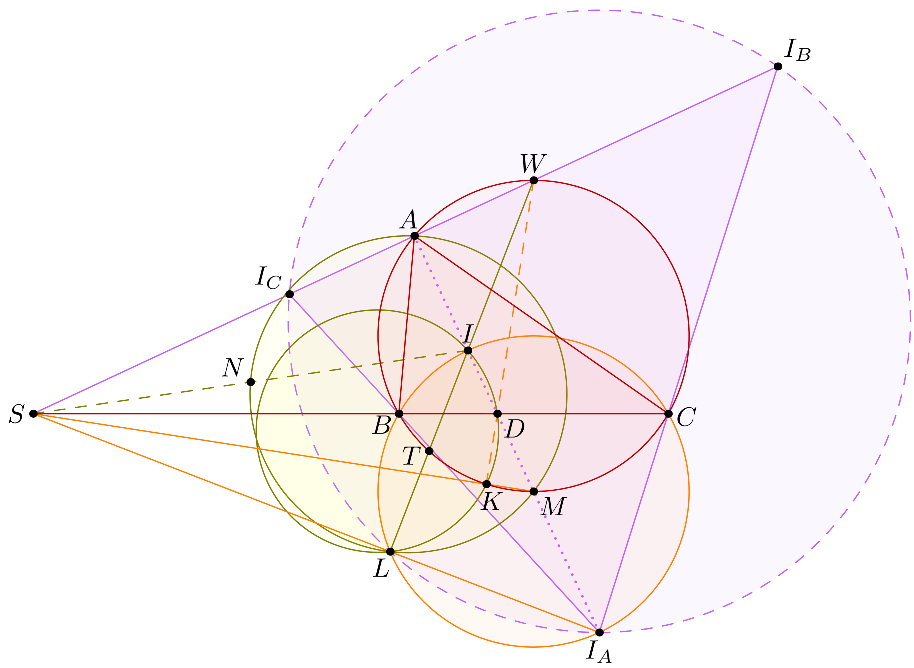
The x^k coefficient of $F(x)$ is the number of permutations having k A-inversions. This is independent of A, so we are done.

§3 USAMO 2017/3 (Evan Chen)

Problem 3 (USAMO 2017/3)

Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

The obvious first step: let W be the midpoint of arc BAC, so W, D, K collinear. By $AB/AC = DB/DC = KB/KC$, we have $-1 = (AK;BC)$, and thus $\overline{AS}$ is the external bisector of $\angle BAC$.



To prove the problem, we will describe a point L with the three required properties: (i) the midpoint of $\overline{IL}$ lies on (ABC), (ii) L lies on (MAN), and (iii) L lies on (KID).

Let $\overline{WI}$ intersect (ABC) again at T, and let L be the reflection of I over T. By design L obeys condition (i).

To prove condition (ii), I contend M, A, N, L lie on the nine-point circle of $\triangle I_A SW$. Note that $\angle ILI_A = \angle ITM = 90^\circ$, so $\overline{I_A A} \perp \overline{WS}$ and $\overline{WL} \perp \overline{I_A S}$. It follows that I is the orthocenter of $\triangle I_A SW$. Then the hypothesis in (ii) becomes clear.

Finally to verify (iii), recall that $\overline{WB}$ is tangent to (BIC), thus $WI \cdot WL = WB^2 = WD \cdot WK$ by Shooting lemma. This completes the proof.

Remark. This is really an orthocenter problem in terms of $\triangle I_A I_B I_C$, with orthic triangle $\triangle ABC$. The desired point L is the so-called "Queue point" of $\triangle I_A I_B I_C$.

§4 USAMO 2017/4 (Maria Monks)

Problem 4 (USAMO 2017/4)

Let P_1, P_2, $\ldots$, P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, none of which is $(1, 0)$. Each point is colored either red of blue, with exactly n red points and n blue points. Let R_1, R_2, $\ldots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 traveling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points B_1, $\ldots$, B_n.

Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering R_1, $\ldots$, R_n of the red points.

First solution, by swapping adjacent points For any $1 \le i < n$, we will consider what happens when we swap the red points R_i, R_{i+1}.

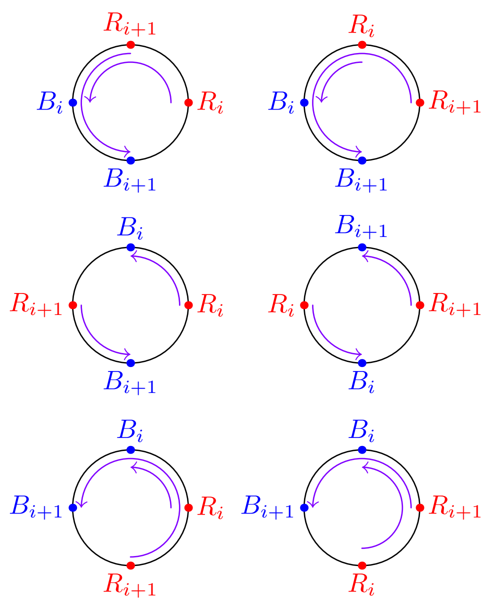
Claim. If we swap R_i, R_{i+1}, then the new arcs R_iB_i, $R_{i+1}B_{i+1}$ will cover the same set of points with multiplicity.

Proof. Delete all the other points. There are three possible ways to orient the four remaining points R_i, R_{i+1}, B_i, B_{i+1}.

Case 1:

Case 2:

Case 3:



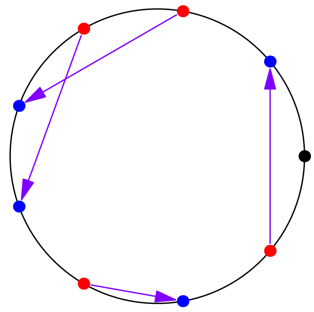
Observe that in all three cases above, the claim holds. $\square$

It suffices to check that for every permutation of $(1, \ldots, n)$, there is a sequence of moves in which we swap adjacent elements that results in the original ordering $(1, \ldots, n)$. This can be done via bubble sort, so we are done.

Second solution, by deleting a chord Start from $(1, 0)$, and progress counterclockwise around the unit circle. Keep a running counter x, beginning at 0. Increment it whenever we cross a blue point, and decrement it whenever we cross a red point.

Say an arc R_iB_i is *good* if it contains $(1, 0)$, and *bad* otherwise. The following claim is independent of the labeling of the red points, so it will suffice:

Claim. The number of good arcs is the maximum value of x.



Reverse induct on n by deleting R_1 and B_1. The base case $n = 1$ is obvious. First note that arc R_1B_1 contains no blue points. There are two cases to consider:

- Suppose $\widehat{R_1B_1}$ is bad. The maximum value of x occurs immediately after crossing a blue point, so there is no change to x.
- Suppose $\widehat{R_1B_1}$ is good. There are no blue points between $(1, 0)$ and B_1, nor are there any between R_1 and $(1, 0)$, so the maximum value of x occurs on the counterclockwise arc B_1R_1. Thus deleting R_1 and B_1 decreases x by 1.

This proves the claim.

§5 USAMO 2017/5 (Ricky Liu)

Problem 5 (USAMO 2017/5)

Find all real numbers $c > 0$ such that there exists a labeling of the lattice points $(x, y) \in \mathbb{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and
- for each label i, the distance between any two points labeled i is at least c^i.

The answer is $c < \sqrt{2}$. We will describe a labeling for each $c < \sqrt{2}$, and then show $c = \sqrt{2}$ (and hence $c \geq \sqrt{2}$) does not work.

Construction for $c < \sqrt{2}$: Suppose that k is the smallest integer such that $c^k < (\sqrt{2})^{k-1}$. Toss on the complex plane, and denote

$$S_1 = \{z : \operatorname{Re}(z) + \operatorname{Im}(z) \equiv 1 \pmod 2\}.$$

Now, let

$$S_t = \{z(1+i) : z \in S_{t-1}\}$$

for all $1 < t < k$. Label all points in S_t the label t, and label the rest of the points the label k. It is easy to see that each point is in exactly one of S_1, S_2, ..., S_k, so this labeling works.

Proof for $c = \sqrt{2}$: We will show that no labeling exists. First we prove a lemma.

Lemma

It is impossible to place four points in the interior of a unit square such that any two points are a distance of at least 1 apart.

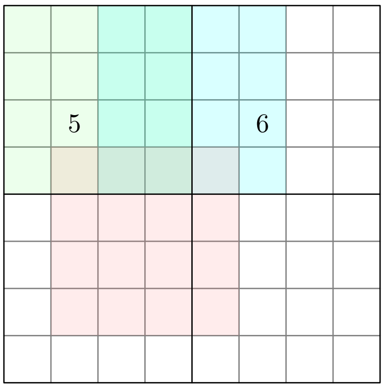
Proof. Consider any four points in the interior of the unit square, and let O be the center of the unit square. By the Pigeonhole Principle there are two points P and Q such that $\angle POQ \leq 90°$. Then $PQ^2 \leq OP^2 + OQ^2 < 1$, as desired. □

Next we claim the following, which obviously implies the desired result.

Claim. Any square of size $2^n \times 2^n$ must contain a cell with label greater than $2n$.

We induct on n, with base case $n = 1$ obvious. Now suppose the claim is true for $n - 1$, and assume for contradiction that it is possible to cells the points of a $2^n \times 2^n$ square such that no label exceeds $2n$.

By the lemma, there are at most three cells labeled $2n$; hence without loss of generality the top-left $2^{n-1} \times 2^{n-1}$ grid does not contain a cell labeled $2n$. By the inductive hypothesis, it must contain a cell labeled $2n - 1$.



Consider the grid of dimensions $2^{n-1} \times 2^{n-1}$ directly to the right of the point labeled $2n-1$, with top row coinciding with the top row of the $2^n \times 2^n$ grid. Clearly this grid cannot contain a $2n-1$, so it contains a $2n$; furthermore this $2n$ is in the top-right $2^{n-1} \times 2^{n-1}$ grid. Then it is easy to construct a grid of dimensions $2^{n-1} \times 2^{n-1}$ orthogonally adjacent to the cell labeled $2n-1$ and either orthogonally or diagonally adjacent to the square labeled $2n$. This grid contains neither a $2n-1$ nor a $2n$, contradiction.

§6 USAMO 2017/6 (Titu Andreescu)

Problem 6 (USAMO 2017/6)

Find the minimum possible value of

$$\frac{a}{b^3+4}+\frac{b}{c^3+4}+\frac{c}{d^3+4}+\frac{d}{a^3+4},$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$.

The key is the tangent-line approximation

$$\frac{x^3}{x^3+4}\le\frac{x}{3}\iff 0\le x(x+1)(x-2)^2.$$

By AM-GM, $ab+bc+cd+da=(a+c)(b+d)\le 4$, so

$$\sum_{\text{cyc}}\frac{a}{b^3+4}=\sum_{\text{cyc}}\frac{a}{4}\left(1-\frac{b^3}{b^3+4}\right)\ge 1-\sum_{\text{cyc}}\frac{ab}{12}\ge\frac{2}{3},$$

with equality achieved by $(a,b,c,d)=(2,2,0,0)$.