USAMO 2016

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§0 Problems

Problem 1. Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S; that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$ for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

Problem 2. Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

Problem 3. Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines I_BF and I_CE meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

Problem 5. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of lines MN and PQ. Denote by ℓ the angle bisector of $\angle MSQ$.

Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter and I is the incenter of triangle ABC.

Problem 6. Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard.

The wizard has $2n$ cards; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order.

You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds.

For which values of n and k is the game winnable?

§1 USAMO 2016/1

Problem 1 (USAMO 2016/1)

Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S; that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$ for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

The answer is 8. We first show that 8 works. We inductively construct a sequence of size $2^{n-1}+1$ where $n = |S|$ and $n \geq 4$, and assume WLOG that $S = \{1, 2, \ldots, n\}$. Start with this construction for $n = 4$:

$$34 \quad 1 \quad 23 \quad 4 \quad 12 \quad 3 \quad 14 \quad 2 \quad 13.$$

At every step, delete the last element, append the resulting sequence to itself, and add an $n+1$ in the middle. Then append $n+1$ to every other element. For instance, $n = 5$ gives

$$\begin{array}{ccccccccc} 345 & 1 & 235 & 4 & 125 & 3 & 145 & 2 & 5 \\ 34 & 15 & 23 & 45 & 12 & 35 & 14 & 25. & \end{array}$$

This obviously works. Now assume for the sake of contradiction $n = 7$ works. It will be immediate that no n less than 7 work. First note that sets of size at least 4 cannot be adjacent to sets of size at least 3, so at most $\binom{7}{1} + \binom{7}{2}$ sets in the sequence have size 4 or more. Hence the length of the sequence cannot exceed

$$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{1} + \binom{7}{2} = 91,$$

a contradiction.

§2 USAMO 2016/2

Problem 2 (USAMO 2016/2)

Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

The key lemma is this:

Lemma

For all k and n,

$$\sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{n} \right\rfloor - \left\lfloor \frac{j}{n} \right\rfloor \right) \leq \left\lfloor \frac{k^2}{n} \right\rfloor .$$

Proof. Rewrite this as

$$\left\lfloor \frac{k^2}{n} \right\rfloor + \sum_{j=0}^{k-1} \left\lfloor \frac{j}{n} \right\rfloor > -1 + \sum_{j=0}^{k-1} \left\lfloor \frac{j+k}{n} \right\rfloor .$$

Noting that equality holds if we erase the floor symbols, we rewrite the inequality using the fractional part: Denote $\{x\} = x - \lfloor x \rfloor$ to obtain

$$\left\{ \frac{k^2}{n} \right\} + \sum_{j=0}^{k-1} \left\{ \frac{j}{n} \right\} < 1 + \sum_{j=0}^{k-1} \left\{ \frac{j+k}{n} \right\} .$$

This obviously holds, as $\{k^2/n\} < 1$ and the sum of the remainders of $0, 1, \dots, k-1$ upon division by n cannot exceed the sum of the remainders of $k, k+1, \dots, 2k-1$ (since the former is the smallest possible sum of k consecutive remainders modulo n). $\square$

By Legendre's Formula, for all primes p,

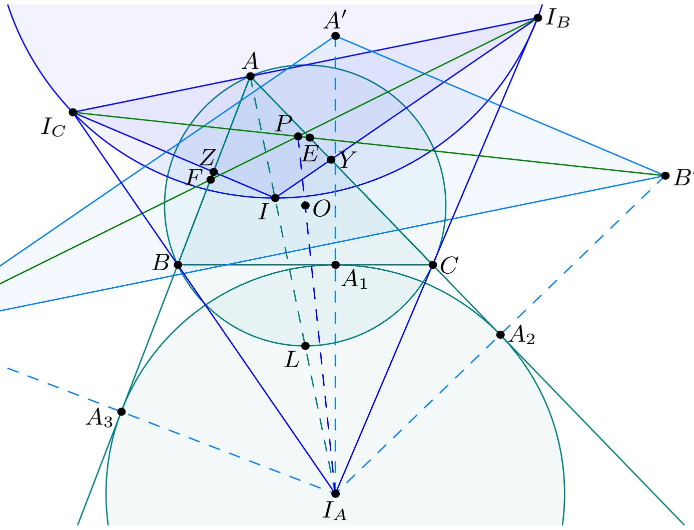
$$\nu_p \left(\prod_{j=0}^{k-1} \frac{(j+k)!}{j!} \right) = \sum_{t=1}^{\infty} \left(\sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{p^t} \right\rfloor - \left\lfloor \frac{j}{p^t} \right\rfloor \right) \right) \leq \sum_{t=1}^{\infty} \left\lfloor \frac{k^2}{p^t} \right\rfloor = \nu_p\Big((k^2)!\Big).$$

This completes the proof.

§3 USAMO 2016/3 (Evan Chen, Telv Cohl)

Problem 3 (USAMO 2016/3)
Let ABC be an acute triangle and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines I_BF and I_CE meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.



Let I be the incenter and I_A the A-excenter of $\triangle ABC$. Denote by A', B', C' the reflections of I_A across $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively, and A_1, B_2, C_3 the projections. The key claim is this:

Claim. Points D, I, A' are collinear, and so are B', E, I_C and C', F, I_B.

Proof. It is well-known that $\overline{IA_1}$ bisects $\overline{AD}$. Since A, I, I_A are collinear, so are D, I, A'. Now, we will show that $\overline{I_CA_2}$ bisects $\overline{BE}$, whence the desired collinearity (B', E, I_C) follows from a similar argument to above.

Let C_2 be the projection of I_C onto $\overline{AC}$, so that $\overline{I_AA_2} \parallel \overline{I_CC_2}$, and set $K = \overline{I_CA_2} \cap \overline{I_AC_2}$. By the homothety centered at B sending the A- to C-excircle,

$$\frac{BI_A}{BI_C} = \frac{I_AA_2}{I_CC_2} = \frac{KA_2}{KI_C},$$

from which $\overline{BK} \parallel \overline{I_AA_2}$, and thus $\overline{BK} \perp \overline{AA_2}$. Now, if $S = \overline{AC} \cap \overline{I_AI_C}$ we may conclude by properties of trapezoid $I_AA_2C_2I_C$ that K is the midpoint of $\overline{BE}$.

By symmetry, our claim has been proven.[1] □

Since $I_AB' = 2I_AB_0 = 2I_AA_0 = I_AA'$ and $CB' = CI_A = CA'$, $\overline{A'B'} \perp \overline{I_AC}$, whence $\overline{A'B'} \parallel \overline{II_C}$. Similarly $\overline{A'C'} \parallel \overline{II_B}$ and $\overline{B'C'} \parallel \overline{I_BI_C}$, thus $\triangle A'B'C'$ and $\triangle II_BI_C$ are homothetic with center P. It follows that P lies on line OI_A.

However, $\overline{OI_A}$ is the Euler line and $\overline{YZ}$ the orthic axis of $\triangle II_BI_C$. It is well-known that they are perpendicular, whence $\overline{PO} \perp \overline{YZ}$. This completes the proof.

[1] An alternate proof is to notice that $-1 = (B, \overline{AC} \cap \overline{I_AI_C}; I_C, I_A) \stackrel{B_0}{=} (B, E; \overline{BE} \cap \overline{A_2I_C}, \infty_{\perp AC})$.

§4 USAMO 2016/4

Problem 4 (USAMO 2016/4)
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

The answer is $f \equiv 0$ and $f(x) \equiv x^2$. Let $P(x, y)$ denote the assertion.

Claim 1. $f(0) = 0$.

Proof. $P(0, 0)$ gives $2f(0)^2 = f(0)^2$, and thus $f(0) = 0$. □

Claim 2. f is even.

Proof. $P(0, y)$ gives $f(y)f(-y) = f(y)^2$ and $P(0, -y)$ gives $f(y)f(-y) = f(-y)^2$. If $f(y) = 0$, then $f(-y) = 0$ and the claim is true. Otherwise $f(-y) = f(y)$, as desired. □

Claim 3. For all x, either $f(4x) = 0$ or $f(x) = x^2$.

Proof. $P(x, -x)$ gives $f(4x)\left[f(x) + f(-x) - 2x^2\right] = 0$. But recall that $f(x) = f(-x)$, so if $f(4x) \neq 0$, then $f(x) = x^2$. □

Claim 4. $f(x) = 0 \iff f(2x) = 0$. In particular, $f(x) = 0$ or $f(x) = x^2$ for all x.

Proof. Suppose $x \neq 0$. $P(x, x)$ gives $2\left(f(x) + x^2\right)f(2x) = f(2x)^2$. If $f(x) = 0$ but $f(2x) \neq 0$, then $8x^4 = 16x^4$, which is absurd If $f(2x) = 0$, then $P(\frac{3}{4}x, \frac{1}{4}x)$ gives

$$f(x)^2 = f(2x)\left(f\left(\frac{x}{4}\right) + \frac{3x^2}{16}\right) = 0,$$

as desired. Thus $f(x) = x^2 \iff f(4x) \neq 0 \iff f(x) \neq 0$. □

Claim 5. If there exists $a \neq 0$ with $f(a) = 0$, then $f \equiv 0$.

Proof. Note that $f(x) \geq 0$ for all x, and furthermore by Claim 2 we can assume $a > 0$. If $z > 0$, $P(\frac{1}{2}a, -\frac{1}{2}z)$ gives

$$0 \leq f\left(\frac{a - z}{2}\right)^2 = \left(f\left(\frac{a}{2}\right) - \frac{az}{2}\right)f(z) = -\frac{az}{2}f(z).$$

But $-\frac{1}{2}az < 0$, so $f(z) = 0$, and by Claim 2 $f \equiv 0$, as desired. □

It is easy to check that $f \equiv 0$ and $f(x) \equiv x^2$ work, so we are done.

§5 USAMO 2016/5

Problem 5 (USAMO 2016/5)

An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of lines MN and PQ. Denote by ℓ the angle bisector of $\angle MSQ$.

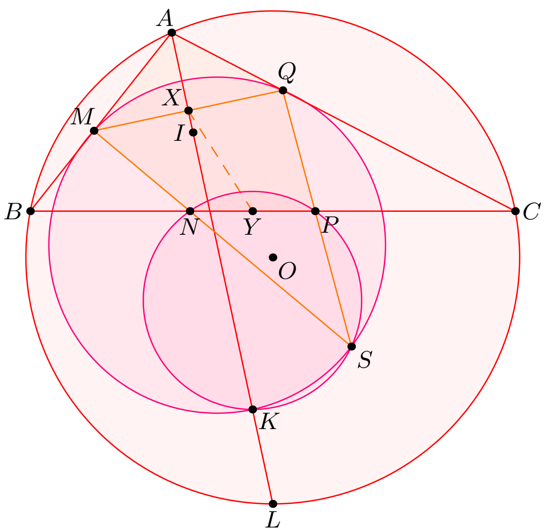
Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter and I is the incenter of triangle ABC.

First solution, by complex numbers Let X, Y, Z be the midpoints of arcs BC, CA, AB not containing A, B, C on the circumcircle of $\triangle ABC$. Toss on the complex plane, with the circumcircle of $\triangle ABC$ as the unit circle, so that $x+y+z$ denotes the incenter. We just need to show that $x+y+z$ is in the direction perpendicular to the external angle bisector of $\angle MSQ$, but since $MN = PQ$, the external angle bisector of $\angle MSQ$ is just

$$(m-n)+(p-q) = (m-a)+(p-n)+(a-q) = ti(x+y+z),$$

where $t = AM = NP = QA$. This completes the proof.

Second solution, by spiral similarity Let $K = (SMQ) \cap (SNP)$ be the Miquel point of $MNPQ$, and let X and Y denote the midpoints of $\overline{MQ}$ and $\overline{MP}$ respectively. Let $\overline{AI}$ intersect (ABC) again at the arc midpoint L, so that $LB = LI = LC$ by the Incenter-Excenter lemma.



Since $MN = PQ$, the spiral similarity at K sending $\overline{MN}$ to $\overline{QP}$ is a congruence; id est we have $\triangle KMN \cong \triangle KQP$. Furthermore $KM = KQ$ and $KN = KP$, so $\overline{KX} \perp \overline{MQ}$ and $\overline{KY} \perp \overline{NP}$.

Note that the antipode of K on (SMQ) lies on ℓ, whence $\measuredangle KXY = \measuredangle KMN = \measuredangle KMS = \measuredangle(\overline{KX}, \ell)$. Thus $\overline{XY} \parallel \ell$, so we only need to prove $\overline{XY} \parallel \overline{IO}$. To do this, I contend $\triangle KXY \sim \triangle LIO$.

Recall A, X, I, K, L are collinear, so the similarity is a homothety and is sufficient. The similarity follows from $\overline{KY} \parallel \overline{LO}$ and

$$\frac{KX}{KY} = \frac{MQ}{NP} = \frac{MQ}{AQ} = \frac{BL}{OL} = \frac{IL}{OL}$$

since $\triangle AMQ \sim \triangle OBL$. This completes the proof.

§6 USAMO 2016/6

Problem 6 (USAMO 2016/6)

Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard.

The wizard has $2n$ cards; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order.

You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is *winnable* if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds.

For which values of n and k is the game winnable?

The answer is $n > k$. First assume that $n > k$, and point to cards 1 to k, 2 to $k+1$, and so on, up to $2n-k+1$ to $2n$. Note that if any of the intervals contains a repeat element, we are done. Otherwise, we now know the cards in indices 1 through $2n-k$. But $2n-k > n$, so two of these are the same, and the game is winnable.

Now assume that $n = k$, and suppose that $S = \{1, 2, \ldots, 2n\}$ is the universal set. At any step after the first, we know a partitioning of S into two sets A and A^c of size n such that no two cards with indices in A have the same label. We know nothing about the orders, so if we pick $X \subseteq A$ and $Y \subseteq A^C$, and point to $X \cup Y$, then it is possible that the cards we pointed to have labels $1, 2, \ldots, n$, and we are in the same exact scenario as before ($A \mapsto X \cup Y$). Hence the game is not winnable, as it is determined purely by the wizard's actions.