USAMO 2015

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Last updated June 12, 2020

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Problem 1. Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

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As X varies on segment PQ, show that M moves along a circle.

Problem 3. Let $S = \{1, 2, \ldots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

Problem 4. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i, k), (i, l), (j, k), (j, l) for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively, or removing one stone from each of (i, l) and (j, k) are moving them to (i, k) and (j, l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

Problem 5. Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

Problem 6. Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \leq n\}$. Assume that for every $n \in \mathbb{N}$, the multiset A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$.

§1 USAMO 2015/1 (Titu Andreescu)

Problem 1 (USAMO 2015/1)
Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

The answer is

$$\boxed{(k^3 - 3k + 1, -k^3 + 3k^2 - 1)} \ \forall k \in \mathbb{Z},$$

and permutations.

Let $a = x + y$ and $b = x - y$, so that

$$\frac{3a^2 + b^2}{4} = x^2 + y^2 + xy = \left(\frac{x+y}{3} + 1\right)^3 = \left(\frac{a}{3} + 1\right)^3.$$

Hence, $3 \mid a$, so let $c = \frac{a}{3}$. It follows that

$$b^2 = 4(c+1)^3 - 27c^2 = 4c^3 - 15c^2 + 12c + 4 = (c-2)^2(4c+1),$$

so $4c + 1$ is an odd perfect square; let $(2k-1)^2 = 4c + 1$. Then $c = k^2 - k$ and $\pm b = 2k^3 - 3k^2 - 3k + 2$. Solving,

$$\{x, y\} = \{k^3 - 3k + 1, -k^3 + 3k^2 - 1\},$$

which work.

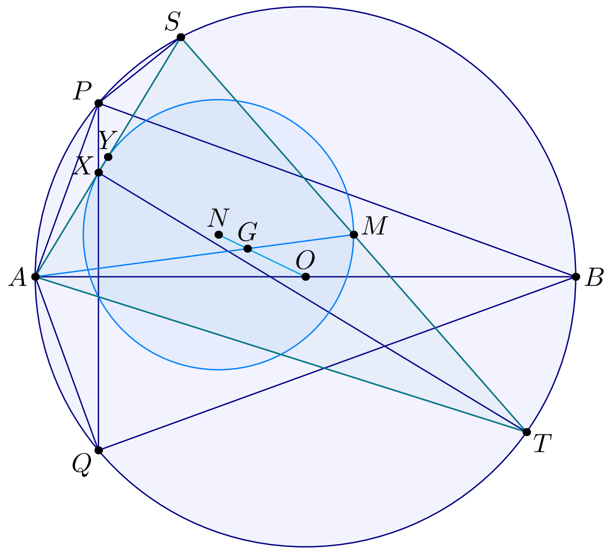
§2 USAMO 2015/2 (Zuming Feng)

Problem 2 (USAMO 2015/2)

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment PQ. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

As X varies on segment PQ, show that M moves along a circle.

Let N, G, O denote the nine-point center, centroid, and circumcenter of $\triangle AST$, and let Y be the midpoint of $\overline{AS}$. Note that (MXY) is the nine-point circle of $\triangle AST$.



Since

$$\measuredangle APX = \measuredangle APQ = \measuredangle PQA = \measuredangle PSA,$$

$\triangle APX \sim \triangle ASP$, so $AP^2 = AX \cdot AS$. Then,

$$AN^2 - \left(\frac{1}{2}AO\right)^2 = \operatorname{Pow}(A, (MXY)) = AX \cdot AY = \frac{1}{2}AX \cdot AS = \frac{1}{2}AP^2.$$

Since AO and AP are fixed, so is AN, whence N moves along a circle centered at A.

A homothety at O with scale factor $\frac{2}{3}$ sends the locus of N to the locus of G, and a homothety at A with scale factor $\frac{3}{2}$ sends the locus of G to the locus of M. Hence, M lies on a circle.

§3 USAMO 2015/3

Problem 3 (USAMO 2015/3)

Let $S = \{1, 2, \ldots, n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be colored red or blue. (The subset itself is assigned a color and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colorings that satisfy the following condition: for any subsets T_1 and T_2 of S,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

The answer is $3^n + 1$. All colorings (except everything red) are of the form: for some sets $A \subseteq B$, color S blue if and only if $A \subseteq S \subseteq B$. It is easy to see that these work, say by PIE with $f(T) = 2^{|T \cap (B \setminus A)|}$.

We will proceed by induction. The base case can be easily verified. Now color subsets without n and subsets with n independently. Suppose a subset in the former category is colored if and only if $A_1 \subseteq S \subseteq B_1$ and a subset in the latter category is colored if and only if $A_2 \cup \{n\} \subseteq S \subseteq B_2 \cup \{n\}$. The goal is to prove $A_1 = A_2$ and $B_1 = B_2$.

The key is to note

$$f(A_1)f(A_2 \cup \{n\}) = f(A_1 \cap A_2)f(A_1 \cup A_2 \cup \{n\}).$$

The left-hand expression is nonzero, so $f(A_1 \cap A_2) \neq 0$ and $A_1 \subseteq A_2$. Note that

$$f(A_2)f(A_1 \cup \{n\}) = f(A_1)f(A_2 \cup \{n\}).$$

Assume for contradiction $A_1 \subset A_2$. Then $f(A_1) = f(A_1 \cup \{n\})$, so $f(A_2) = f(A_2 \cup \{n\})$, absurd. Analogously $B_1 = B_2$, so we are done.

§4 USAMO 2015/4

Problem 4 (USAMO 2015/4)
Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can perform *stone moves*, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions (i,k), (i,l), (j,k), (j,l) for some $1 \leq i,j,k,l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) are moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

Let the m stones have coordinates (a_1, b_1), (a_2, b_2), $\ldots$, (a_m, b_m). The *signature* (A, B) of the piling is defined by the multisets $A = \{a_i : 1 \leq i \leq m\}$ and $B = \{b_i : 1 \leq i \leq m\}$. Note that under any stone move, the signature is invariant. Furthermore, each signature determines at least one way of piling the stones: denote by $a_1 \leq a_2 \leq \cdots \leq a_m$ and $b_1 \leq b_2 \leq \cdots \leq b_m$ the elements of A and B and let the stones occupy squares (a_1, b_1), (a_2, b_2), $\ldots$, (a_m, b_m).

I contend that two pilings with the same signature are equivalent. Indeed, each stone move sends the points (a_i, b_i) and (a_j, b_j) to (a_i, b_j) and (a_j, b_i) for some i and j. Thus we are merely swapping b_i and b_j. Since each permutation is obtained by a series of swaps, each signature determines an equivalence class.

Finally, by Stars and Bars there are $\binom{m+n-1}{m}^2$ possible signatures.

§5 USAMO 2015/5 (Mohsen Jamali)

Problem 5 (USAMO 2015/5)

Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

Assume for contradiction $p = ac + bd$ is prime, and assume without loss of generality $a < c$. Then

$$\begin{aligned} p \mid ac + bd \mid a^4c^4 - b^4d^4 \mid a^4\left(e^5 - d^4\right) - d^4\left(e^5 - a^4\right) \\ = e^5\left(a^4 - d^4\right) = e^5(a-d)(a+d)\left(a^2 + d^2\right). \end{aligned}$$

However note that:

- $p > e$, since $p^5 = (ac+bd)^5 \geq (a+b)^5 \geq a^5 + b^5 \geq a^4 + b^4 = e^5$, where equality never holds since a, b, c, d distinct.
- $p > a + d$, since $ac + bd \geq a + d$, where equality never holds since a, b, c, d distinct.

Hence $p \mid a^2 + d^2$. This means $ac + bd \leq a^2 + d^2$, or $a(c-a) \leq d(d-b)$. Recall that $a < c$, so $b < d$ as well. This implies $a^4 + b^4 < c^4 + d^4$, which is absurd.

Remark. One solution is $37483800763^4 + 100380347806^4 = 84497381381^4 + 85132700038^4 = 6353186575^5$. To generate this solution, we use the well-known equality $59^4 + 158^4 = 133^4 + 134^4$. We let the common value be x. Then $(59x)^4 + (158x)^4 = (133x)^4 + (134x)^4 = x^5$ works.

§6 USAMO 2015/6

Problem 6 (USAMO 2015/6)

Consider $0 < \lambda < 1$, and let A be a multiset of positive integers. Let $A_n = \{a \in A : a \le n\}$. Assume that for every $n \in \mathbb{N}$, the multiset A_n contains at most $n\lambda$ numbers. Show that there are infinitely many $n \in \mathbb{N}$ for which the sum of the elements in A_n is at most $\frac{n(n+1)}{2}\lambda$.

Let a_n be the multiplicity of n, and let $y_n = a_1 + \cdots + a_n$ and $x_n = n\lambda - y_n > 0$. Assume for contradiction that

$$x_n < \frac{x_1 + \cdots + x_{n-1}}{n} \quad \text{for all } n > N$$

for some large integer N.

Claim. Showing this is impossible solves the problem.

Proof. Indeed, this implies that

$$\sum_{k=1}^{n} k a_k = (n+1)y_n - \sum_{k=1}^{n} y_k = \frac{n(n+1)}{2}\lambda - (n+1)x_n + \sum_{k=1}^{n} x_k > \frac{n(n+1)}{2}\lambda$$

for sufficiently large n, contradiction. □

Denote $\varepsilon = \min(\lambda, 1-\lambda)$. It is clear that $|x_n - x_{n-1}| \ge \varepsilon$ for all n. Say that M is the maximum value of x_n, since the sequence is clearly bounded. Then since $\frac{1}{2}(x_n + x_{n+1}) < M - \frac{1}{2}\varepsilon$ for all n, there is a large integer N_1 such that

$$x_n < \frac{x_1 + \cdots + x_{n-1}}{n} < M - \frac{\varepsilon}{3} \quad \text{for all } n > N_1.$$

(Note that $\frac{1}{3}\varepsilon$ is just an arbitrary constant smaller than $\frac{1}{2}\varepsilon$.) Now M decreases by $\frac{1}{3}\varepsilon$ every time we apply this argument, so eventually $M < 0$, contradiction.