USAMO 2014

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Last updated June 12, 2020

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§0 Problems

Problem 1. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

Problem 2. Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ such that $x \neq 0$.

Problem 3. Prove that there exists an infinite set of points

$$\dots, P_{-3}, P_{-2}, P_{-1}, P_0, P_1, P_2, P_3, \dots$$

in the plane with the following property: for any three distinct integers a, b, c, points P_a, P_b, P_c are collinear if and only if $a + b + c = 2014$.

Problem 4. Let k be a positive integer. Alex and Bob play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with Alex moving first. In his move, Alex may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, Bob may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, Alex wins. Find the minimum value of k for which Alex cannot win in a finite number of moves, or prove that no such minimum value exists.

Problem 5. Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of $\triangle AHC$ with the internal bisector of $\angle BAC$. Let X be the circumcenter of $\triangle APB$ and Y the orthocenter of $\triangle APC$. Prove that the length of segment XY is equal to the circumradius of $\triangle ABC$.

Problem 6. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a + i, b + j) > 1$ for all $i, j \in \{0, 1, \dots, n\}$, then

$$\min\{a, b\} > (cn)^{n/2}.$$

§1 USAMO 2014/1

Problem 1 (USAMO 2014/1)

Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2+1)(x_2^2+1)(x_3^2+1)(x_4^2+1)$ can take.

The answer is 16. Let $i = \sqrt{-1}$. The key observation is that $x^2+1 = (-i-x)(i-x)$; expressing the relevant product in this form, we have

$$\begin{aligned}\prod_{k=1}^{4}\left(x_k^2+1\right) &= \prod_{k=1}^{4}(-i-x_k)(i-x_k) \\ &= P(-i)P(i) = |P(i)|^2 \\ &= |(1-b+d)+(c-a)i| \\ &= (1-b+d)^2+(c-a)^2 \\ &\geq (1-5)^2 = 16.\end{aligned}$$

Equality holds when $x_1 = x_2 = x_3 = x_4 = 1$.

§2 USAMO 2014/2

Problem 2 (USAMO 2014/2)
Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y)-x)+y^2f(2x-f(y))=\frac{f(x)^2}{x}+f(yf(y))$$

for all $x, y \in \mathbb{Z}$ such that $x \neq 0$.

The answers are $f \equiv 0$ and $f(x) \equiv x^2$, which work. Let $P(x, y)$ denote the assertion.

Claim 1. $f(0) = 0$.

Proof. Let p be any prime. By setting $x = p$, it is clear $p \mid f(p)$, and by $P(p, 0)$, we have $f(0) \equiv 0 \pmod{p}$ for all primes p. The claim follows. $\square$

Claim 2. For all n, either $f(n) = f(-n) = 0$ or $f(n) = f(-n) = n^2$.

Proof. By $P(n, 0)$, we have $n^2 f(-n) = f(n)^2$, and analogously by $P(-n, 0)$, we have $n^2 f(n) = f(-n)^2$. The claim should be obvious from here. $\square$

Finally we settle the pointwise trap. Let $f(t) = 0$ while $t \neq 0$; we will show $f \equiv 0$. Set $y = t$ to find

$$xf(x)+t^2f(2x)=\frac{f(x)^2}{x}.$$

In both cases $f(x) = 0$ and $f(x) = x^2$, we have $f(2x) = 0$, so f sends all even integers to zero.

Let $f(s) = s^2$ for odd $s > 0$ (since f is even). Set $x \neq 0$ even and $y = s$ to find

$$s^2f(2x-s^2)=f(s^3).$$

- If $f(s^3) = s^6$, then
$$s^4 \in \left\{0, (2x-s^2)^2\right\} \quad \text{for all } x,$$
which is absurd.
- If $f(s^3) = 0$, then vary x to show f sends all $3 \pmod{4}$ numbers, except potentially $-s^2$, to zero. But f is even, so f sends all odds to zero, except potentially $\pm s^2$. In particular, we have the contradiction $f(s) = 0$ unless $s = 1$. Take $P(5, 1)$ to arrive at a contradiction.

§3 USAMO 2014/3

Problem 3 (USAMO 2014/3)

Prove that there exists an infinite set of points

$$\ldots,\ P_{-3},\ P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2,\ P_3,\ \ldots$$

in the plane with the following property: for any three distinct integers a, b, c, points P_a, P_b, P_c are collinear if and only if $a+b+c=2014$.

The construction is $P_n = \left(n, n^3 - 2014n^2\right)$ for all integers n.

Let $f(x) = x^3 - 2014x^2$. For any reals a, b, c, it holds that $A = (a, f(a))$, $B = (b, f(b))$, $C = (c, f(c))$ are collinear if and only if $a+b+c = 2014$. This can be easily seen via Vieta's formulas: if the line $y = px + q$ passes through $(a, f(a))$, $(b, f(b))$, then the x-coordinates of its intersection with $f(x)$ satisfy $0 = x^3 - 2014x^2 - px - q$, so its third intersection with $f(x)$ is $(2014 - a - b, f(2014 - a - b))$.

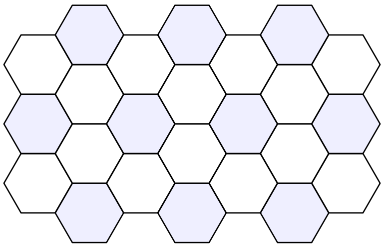
§4 USAMO 2014/4

Problem 4 (USAMO 2014/4)

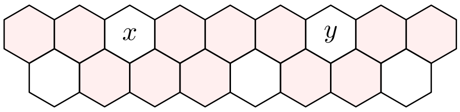
Let k be a positive integer. Alex and Bob play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with Alex moving first. In his move, Alex may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, Bob may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, Alex wins. Find the minimum value of k for which Alex cannot win in a finite number of moves, or prove that no such minimum value exists.

The answer is $k = 6$.

Bob's strategy for $k = 6$: Consider the below "honeycomb" coloring. Bob can ensure that at any point in time, at most one of the blue hexagons is colored; thus the longest line of labeled hexagons has length five.



Alice's strategy for $k = 5$: On the contrary, consider arbitrarily-long two chains of long hexagons in a "parallelogram" formation as follows. Play strictly within the chain until it is no longer possible; that is, until there are no two adjacent uncovered squares.



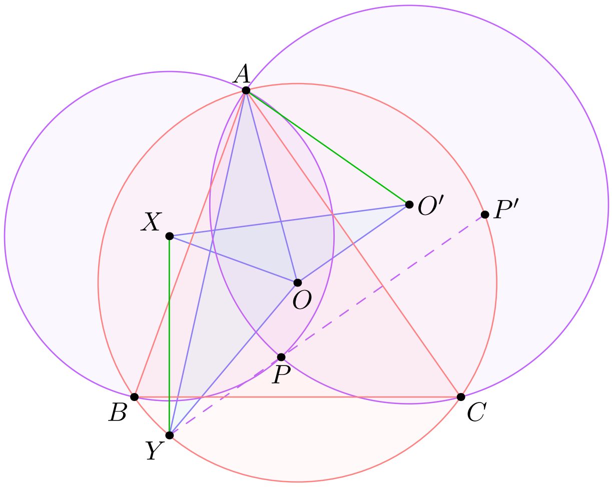
If we have not won already, then there are two uncovered hexagons x, y in the top row with $k \leq 4$ covered hexagons between them.

- If $k = 4$, we just win by covering either x or y.
- If $k \leq 3$, then consider the hexagons in the bottom row adjacent to x and y. By hypothesis, all four are covered, and at most one of the $k - 2 \leq 2$ hexagons between them is uncovered. We can easily win by filling it.

§5 USAMO 2014/5

Problem 5 (USAMO 2014/5)
Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of $\triangle AHC$ with the internal bisector of $\angle BAC$. Let X be the circumcenter of $\triangle APB$ and Y the orthocenter of $\triangle APC$. Prove that the length of segment XY is equal to the circumradius of $\triangle ABC$.

By properties of the orthocenter, (APC) is the reflection of (ABC) over $\overline{AC}$, and the reflection of Y over $\overline{AC}$ lies on (APC), so Y lies on (ABC). Let O' be the circumcenter of $\triangle APC$, i.e. the reflection of O over $\overline{AC}$, and let P' be the reflection of P over $\overline{AC}$, so that P' lies on (ABC) and $\overline{YPP'} \perp \overline{AC}$.



Claim. $\triangle OAY \sim \triangle OO'X$.

Proof. First $\measuredangle OXO' = \measuredangle BAP = \measuredangle PAC = \measuredangle XO'O$, so $OX = OO'$. Furthermore

$$\measuredangle OYA = 90^\circ + \measuredangle PP'A = 90^\circ + \measuredangle APP' = \measuredangle PAC = \measuredangle XO'O,$$

and the claim readily follows. □

Thus O is the center of spiral similarity sending $\overline{AY}$ to $\overline{O'X}$. As spiral similarities come in pairs, $\triangle OXY \sim \triangle OO'A$. Since $OY = OA$, this similarity is a rotation, so $XY = AO' = AO$, as needed.

§6 USAMO 2014/6

Problem 6 (USAMO 2014/6)

Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \ldots, n\}$, then

$$\min\{a, b\} > (cn)^{n/2}.$$

To simplify computation, we only use $i, j \in \{1, \ldots, n\}$. We will prove the stronger bound $\min\{a, b\} > (cn)^n$ for sufficiently large n.

Let $\varepsilon = 10^{-10}$ be small. Consider the $n \times n$ grid defined by the points $(a+i, b+j)$ where $i, j \in \{1, \ldots, n\}$, and in each cell place the least prime factor of $\gcd(a+i, b+j)$. Note that each prime p divides at most $(1 + n/p)^2$ cells of the grid.

Claim. For n large, at most $n^2/2$ of the cells of the grid contain a prime $p < \varepsilon n^2$.

Proof. The number of primes covered is

$$\sum_{p<\varepsilon n^2} \left(1 + \frac{n}{p}\right) \le \pi(\varepsilon n^2) + 2n \sum_{p<\varepsilon n^2} \frac{1}{n} + n^2 \sum_{p<n} \frac{1}{p^2} < \frac{n^2}{2}$$

for sufficiently large n. $\square$

Remark. Some more details on the estimates: where $s = \varepsilon n^2$, we have

$$\pi(s) = \frac{s}{\log s}\left(1 + O\left(\frac{1}{\log s}\right)\right) = o\left(n^2\right)$$
$$\sum_{p<s} \frac{1}{p} < \sum_{k=1}^{s} \frac{1}{p} = O(\log s) = o(n^2)$$
$$\sum_{p<s} \frac{1}{p^2} < \sum_{p} \frac{1}{p^2} \approx 0.452 < \frac{1}{2}.$$

For curiosity sake, the best bound for the second expression is $\sum_{p<s} \frac{1}{p} = \log\log s \cdot (1 + o(1))$.

Remark. Here is an easy proof of $\sum_p \frac{1}{p^2} < \frac{1}{2}$. Note that

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \sum_{n} \frac{1}{n^2} - \sum_{n \text{ even}} \frac{1}{n^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} < \frac{5}{4}$$

by $\pi^2 < 10$. Then

$$\sum_{p} \frac{1}{p^2} < -\frac{1}{1^2} + \frac{1}{2^2} + \sum_{n \text{ odd}} \frac{1}{n^2} < \frac{1}{2}.$$

Hence for some i, the row $a+i$ contains at least $n/2$ primes $p \ge \varepsilon n^2$. For $n > \varepsilon^{-1}$, none of the primes divide two numbers of the form $b+j$, so these $n/2$ primes are all distinct. Then

$$(a+i)^n > (\varepsilon n^2)^{n/2} = \varepsilon^{n/2} \cdot n^n,$$

as needed.

> **Remark.** We prove $\min\{a, b\} > (cn)^n$ instead. The requested bound $(cn)^{n/2}$ is derived by proving primes $p < n$ cover at most $n^2/2$ cells, and using that estimate instead.