USAMO 2010

Compiled by Eric Shen

Last updated June 12, 2020

Contents

§0 Problems

Problem 1. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Problem 2. There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

Problem 3. The 2010 positive real numbers a_1, a_2, $\ldots$, a_{2010} satisfy the identity $a_i a_j \le i + j$ for all $1 \le i < j \le 2010$. Determine, with proof, the largest possible value of the product $a_1 a_2 \cdots a_{2010}$.

Problem 4. Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

Problem 5. Let $q = \frac{3p-5}{2}$, where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

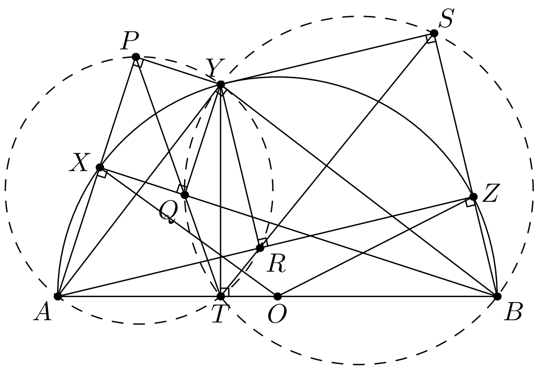
Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

Problem 6. A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k at most one of the pairs (k, k) and $(-k, -k)$ is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.

§1 USAMO 2010/1

Problem 1 (USAMO 2010/1)

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.



Let T be the projection of Y onto $\overline{AB}$. Notice that T lies on the Simson Line $\overline{PQ}$ from Y to $\triangle AXB$, and the Simson Line $\overline{RS}$ from Y to $\triangle AZB$. Hence, $T = \overline{PQ} \cap \overline{RS}$, so it suffices to show that $\angle PTS = \frac{1}{2}\angle XOZ$.

Since $TAPY$ and $TBSY$ are cyclic quadrilaterals,

$$\angle PTS = \angle PTY + \angle YTS = \angle XAY + \angle YBZ = \frac{1}{2}\angle XOY + \frac{1}{2}\angle YOZ = \frac{1}{2}\angle XOZ,$$

as required.

§2 USAMO 2010/2

Problem 2 (USAMO 2010/2)

There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

We say a student *moves* if he swaps with the person in front of him. The key is this estimate:

Claim. The person with height h_k can move at most $\binom{k-1}{2}$ times.

Proof. Induct on k. The base cases $k = 1$ and $k = 2$ are clear. Then if the claim is true for integers less than k, the person with height h_k can move at most

$$\binom{k-2}{2} + (k-1) = \binom{k-1}{2}$$

times; otherwise he must switch with someone of height at least h_{k-1}. □

Finally, the number of moves possible is at most

$$\sum_{k=1}^{n} \binom{k-1}{2} = \binom{n}{3}$$

by Hockey-Stick, thus concluding the proof.

§3 USAMO 2010/3

Problem 3 (USAMO 2010/3)
The 2010 positive real numbers a_1, a_2, $\ldots$, a_{2010} satisfy the identity $a_ia_j \le i+j$ for all $1 \le i < j \le 2010$. Determine, with proof, the largest possible value of the product $a_1a_2\cdots a_{2010}$.

It is the obvious bound $3 \cdot 7 \cdot 11 \cdots 4019$, with upper bound obtained by

$$a_1a_2\cdots a_{2010} = (a_1a_2)\cdots(a_{2019}a_{2020}) \le 3\cdots 4019.$$

I am aware of two constructions, shown below. In both, $a_{2n}a_{2n-1} = 2n-1$, so equality holds.

Remark. Some preliminary remarks on motivation: for constructions, it turns out the most concerning bounds are those where $|i-j|=1$, and overall the bounds are more concerning when i, j increase (hence why we fix $a_{2008}a_{2010}$ in the second construction), so we should push for equality in these cases.

This gives the desired upper bound, and to achieve this, heuristically we want $a_n \approx \sqrt{2n}$. Both the below solutions build upon this estimate.

First, explicit construction Let $a_{2n} = 2\sqrt{n}$ and $a_{2n-1} = 2\sqrt{n} - \frac{1}{2\sqrt{n}}$. We manually verify the required inequality:

- Let $i = 2a$, $j = 2b$, where $a < b$. Then

$$a_ia_j = \left(2\sqrt{a}\right)\left(2\sqrt{b}\right) = 4\sqrt{ab} < 2(a+b).$$

- Let $i = 2a-1$, $j = 2b-1$, where $a < b$. Note the estimate

$$\frac{1}{4\sqrt{ab}} \le \frac{1}{4\sqrt{a(a+1)}} < \frac{2}{\left(\sqrt{a+1}-\sqrt{a}\right)^2} = 2\left(\sqrt{a+1}-\sqrt{a}\right)^2 \le 2\left(\sqrt{b}-\sqrt{a}\right)^2,$$

From this, we have

$$a_ia_j = \left(2\sqrt{a} - \frac{1}{2\sqrt{a}}\right)\left(2\sqrt{b} - \frac{1}{2\sqrt{b}}\right) < 4\sqrt{ab} + \frac{1}{4\sqrt{ab}} - 2 \le 2(a+b) - 2.$$

- Let $i = 2a-1$, $j = 2b$, where $a \le b$. Then

$$a_ia_j = \left(2\sqrt{a} - \frac{1}{2\sqrt{a}}\right)\left(2\sqrt{b}\right) = 4\sqrt{ab} - \sqrt{\frac{b}{a}} \le 2(a+b) - 1.$$

- Let $i = 2a$, $j = 2b-1$, where $a < b$. Note the estimate

$$\sqrt{\frac{1}{b}} = \frac{2}{2\sqrt{b}} \le \frac{2(b-a)}{\sqrt{b}+\sqrt{a}} = 2\left(\sqrt{b}-\sqrt{a}\right) \implies 1 - \sqrt{\frac{a}{b}} = 2\left(\sqrt{b}-\sqrt{a}\right)^2.$$

From this, we have

$$a_ia_j = (2\sqrt{a})\left(2\sqrt{b} - \frac{1}{2\sqrt{b}}\right) = 4\sqrt{ab} - \sqrt{\frac{a}{b}} \le 2(a+b) - 1.$$

Hence $a_ia_j \le i+j$ for all i, j.

Second, more motivated construction Take the sequence with $a_n a_{n+1} = 2n+1$ for each n and $a_{2008}a_{2010} = 4028$. Note that the inequality holds when $j-1=1$ by definition. We first verify the required inequality for $j-i=2$. Backwards induct on i, with the base case $i=2008$ given and the case $i=2007$ clear by noting

$$a_{2007}a_{2009} = \frac{4015}{a_{2018}} \cdot \frac{4017}{a_{2010}} = \frac{4015 \cdot 4017}{4018} \leq 4017.$$

If the inequality holds for integers greater than i, then

$$a_i a_{i+2} = \frac{(a_i a_{i+1})(a_{i+2}a_{i+3})(a_{i+2}a_{i+4})}{(a_{i+1}a_{i+2})(a_{i+3}a_{i+4})} \leq \frac{(2i+1)(2i+5)(2i+6)}{(2i+3)(2i+7)} \leq 2i+2,$$

thus settling $j-i=2$. We now prove the inequality for all i, j, by induction on $j-i$ with increment 2. The base cases $j-i \in \{1,2\}$ have already been proven.

If $a_i a_j \leq i+j$, then

$$a_i a_{j+2} \leq \frac{a_{j+2}}{a_j}(i+j) = \frac{a_{j+1}a_{j+2}}{a_j a_{j+1}}(i+j) = \frac{2j+3}{2j+1}(i+j) \leq i+j+2,$$

where the last inequality holds since

$$\frac{2j+3}{2j+1}(i+j) \leq i+j+2 \iff \frac{2}{2j+1}(i+j) \leq 2 \iff i < j.$$

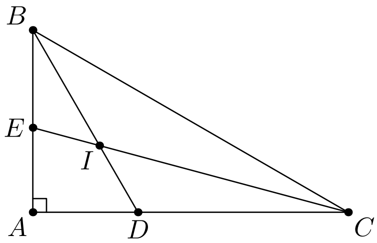
This completes the proof.

§4 USAMO 2010/4

Problem 4 (USAMO 2010/4)

Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

The answer is no. In fact it is not possible for AB, AC, BI, CI to have integer lengths. Assume for contradiction otherwise.



Note that $\angle BIC = 135°$. By the Law of Cosines on $\triangle BIC$,

$$-\frac{1}{\sqrt{2}} = \frac{BI^2 + CI^2 - BC^2}{2 \cdot BI \cdot CI} = \frac{BI^2 + CI^2 - AB^2 - AC^2}{2 \cdot BI \cdot CI} \in \mathbb{Q},$$

contradiction.

§5 USAMO 2010/5

Problem 5 (USAMO 2010/5)
Let $q = \frac{3p-5}{2}$, where p is an odd prime, and let

$$S_q = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m-n$ is divisible by p.

Let $H_n = \sum_{k=1}^n 1/k$. It is not hard to see that

$$\begin{aligned}
S_q &= \sum_{k=1}^{\frac{1}{2}(p-1)} \frac{1}{3k(3k-1)(3k+1)} \\
&= \sum_{k=1}^{\frac{1}{2}(p-1)} \left(\frac{1/2}{3k-1} - \frac{1}{3k} + \frac{1/2}{3k+1}\right) \\
&= \frac{1}{2}\sum_{k=1}^{\frac{1}{2}(p-1)} \left(\frac{1}{3k-1} + \frac{1}{3k} + \frac{1}{3k+1}\right) - \frac{3}{2}\sum_{k=1}^{\frac{1}{2}(p-1)} \frac{1}{3k} \\
&= \frac{1}{2}\left(H_{\frac{1}{2}(3p-1)} - H_{\frac{1}{2}(p-1)} - 1\right),
\end{aligned}$$

whence

$$\frac{m}{n} = \frac{1}{p} - 2S_q = \frac{1}{p} - \left(H_{\frac{1}{2}(3p-1)} - H_{\frac{1}{2}(p-1)} - 1\right) = 1 - \left(\sum_{k=\frac{1}{2}(p+1)}^{\frac{1}{2}(3p-1)} \frac{1}{k}\right) + \frac{1}{p}.$$

Conveniently the $1/p$ cancels out, so taking everything modulo p,

$$\frac{m}{n} \equiv 1 - \sum_{k=1}^{p-1} \frac{1}{k} \equiv 1 - \sum_{k=1}^{p-1} k \equiv 1 \pmod{p},$$

i.e. $m - n \equiv 0 \pmod{p}$, as desired.

§6 USAMO 2010/6

Problem 6 (USAMO 2010/6)

A blackboard contains 68 pairs of nonzero integers. Suppose that for each positive integer k at most one of the pairs (k,k) and $(-k,-k)$ is written on the blackboard. A student erases some of the 136 integers, subject to the condition that no two erased integers may add to 0. The student then scores one point for each of the 68 pairs in which at least one integer is erased. Determine, with proof, the largest number N of points that the student can guarantee to score regardless of which 68 pairs have been written on the board.

Assume without loss of generality all pairs of the form (k,k) obey $k>0$. Otherwise swap all instances of k and $-k$. For each $k>0$, erase all instances of k with probability p, and erase all instances of $-k$ with probability $1-p$. Then, for $k,a,b>0$:

- some element of (k,k) is erased with probability p;
- some element of (a,b), $a \neq b$ is erased with probability $1-(1-p)^2>p$;
- some element of $(a,-b)$ is erased with probability $1-p(1-p)>p$;
- some element of $(-a,-b)$ is erased with probability $1-p^2=p$.

Hence we expect at least $\lceil 68p \rceil = 43$ points.

Now we establish the upper bound. For each $k=1,\ldots,8$, take five instances of (k,k), and for all $1 \le a < b \le 8$, take a single instance of $(-a,-b)$. Thus we have chosen $5 \cdot 8 + \binom{8}{2} = 68$ pairs. It suffices to show at most 43 points can be scored.

Iterate over $k=1,\ldots,8$ and choose whether we erase k or $-k$. At any given point, we have chosen t negative numbers, then we earn $7-t$ points by choosing $-k$, and we earn 5 points by choosing k. Thus the maximum attainable score is $7+6+5+5+5+5+5+5=43$, the end.