USA TST 2021

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§0 Problems

Problem 1. Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

Problem 2. Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D. Let O and ρ denote the circumcenter and circumradius of $\triangle XCD$, respectively.

Prove that there exists a fixed point K and a real number c, independent of X, for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of X.

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x}f(x) + \frac{y-x}{z-x}f(z)\right) \leq f\left(\frac{x+z}{2}\right) - \frac{f(x)+f(z)}{2}$$

for all real numbers $x < y < z$.

§1 USA TST 2021/1 (Ankan Bhattacharya, Michael Ren)

Problem 1 (USA TST 2021/1)

Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a+b+c+d$ and s divides $abc+abd+acd+bcd$.

The answer is non-primes.

Observe that $a+b+c+d \mid abc+abd+acd+bcd$ is equivalent to

$$\begin{aligned} 0 &\equiv abc + (ab+bc+ca)d \\ &\equiv abc - (a+b+c)(ab+bc+ca) \\ &\equiv -(a+b)(b+c)(c+a) \pmod{a+b+c+d}. \end{aligned}$$

Note that $a+b$, $b+c$, $c+a$ are each less than $a+b+c+d$, so the condition cannot hold if $s = a+b+c+d$ is prime. Moreover, each non-prime $s = mn$ can be attained by taking $a = 1$, $b = m-1$, $c = n-1$, and $d = (m-1)(n-1)$, so the answer follows.

Remark (Alternate proof that primes fail)**.** Note that if s is prime, $(x-a)(x-b)(x-c)(x-d)$ will be an even polynomial in $\mathbb{F}_s[x]$, so its roots come in pairs that sum to a multiple of s, specifically $\geq s$, absurd.

§2 USA TST 2021/2 (Andrew Gu, Frank Han)

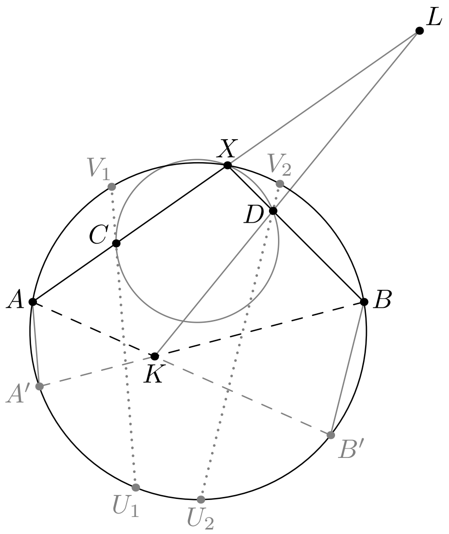
Problem 2 (USA TST 2021/2)

Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D. Let O and ρ denote the circumcenter and circumradius of $\triangle XCD$, respectively.

Prove that there exists a fixed point K and a real number c, independent of X, for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of X.

WLOG, diagram as shown. Let A' and B' lie on Γ such that $\overline{AA'} \parallel \overline{U_1V_1}$ and $\overline{BB'} \parallel \overline{U_2V_2}$. I claim the fixed point is $K = \overline{AB'} \cap \overline{BA'}$.



We will show that $\operatorname{Pow}(K, (XCD))$ depends only on the locations of A, B, K, and the distances α and β from C and D to $\overline{AA'}$ and $\overline{BB'}$, respectively. To this end, define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\bullet) = \operatorname{Pow}(\bullet, (XCD)) - \operatorname{Pow}(\bullet, \Gamma).$$

It is known that f is linear.

First, note that

$$\begin{aligned} \alpha &= AC \sin \angle CAA' = AC \sin \angle XBK \\ \beta &= BD \sin \angle DBB' = BD \sin \angle XAK. \end{aligned}$$

Denote $L = \overline{AX} \cap \overline{DK}$. It is easy to compute that

$$\begin{aligned} f(L) &= LX \cdot LC - LX \cdot LA = -LX \cdot CA \\ f(D) &= -(-DB \cdot XD) = DB \cdot XD. \end{aligned}$$

Moreover, observe from the Law of Sines that

$$\begin{aligned} \frac{KL}{LD} &= \frac{AK \cdot \sin \angle XAK}{LD \cdot \sin \angle XLD} = \frac{AK \cdot \sin \angle XAK}{XD \cdot \sin \angle AXB} \\ \frac{KD}{LD} &= \frac{BK \cdot \sin \angle XBK}{LD \cdot \sin \angle LDX} = \frac{BK \cdot \sin \angle XBK}{LX \cdot \sin \angle AXB}. \end{aligned}$$

Since f is linear, we know

$$\begin{aligned} f(K) &= \frac{KL}{LD} \cdot f(D) - \frac{KD}{LD} \cdot f(L) \\ &= \frac{AK \cdot \sin \angle XAK \cdot DB + BK \cdot \sin \angle XBK \cdot CA}{\sin \angle AXB} \\ &= \frac{AK \cdot \beta + BK \cdot \alpha}{\sin \angle AXB}, \end{aligned}$$

which is dependent only on A, B, K, α, β.

Since $\operatorname{Pow}(K, \Gamma)$ is fixed, $\operatorname{Pow}(K, (XCD))$ is also dependent only on A, B, K, α, β, so we are done.

§3 USA TST 2021/3 (Gabriel Carroll)

Problem 3 (USA TST 2021/3)
Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z)\right) \le f\left(\frac{x+z}{2}\right) - \frac{f(x)+f(z)}{2}$$

for all real numbers $x < y < z$.

The answer is linear functions and downward-facing parabolas, which can easily be shown to work. Observe that equality always holds for linear f, so we are free to shift f by any linear function while preserving the given condition; moreover, we can scale f by any positive constant.

It is easy to rearrange the given equation into the following form:

Claim 1 (Rewritten FE). For all $x < y < z$,

$$\frac{f(z)-f(x)}{z-x} - \frac{f(y) - f\left(\frac{x+z}{2}\right)}{y - \frac{x+z}{2}}$$

equals 0 or has the same sign as $y - \frac{x+z}{2}$.

We will call the above assertion $P(x, y, z)$.

First, I contend:

Claim 2 (Continuity and concavity). f is continuous at every point; moreover, for all x, z, we have

$$f\left(\frac{x+z}{2}\right) \ge \frac{f(x)+f(z)}{2}.$$

Proof. Assume by shifting that $f(-1) = f(1) = 0$. We will show that $f(0) \ge 0$, and that f is continuous at 0.

First, applying $P(-1, y, 1)$ gives $f(y) \le f(0)$ for all $-1 < y < 1$. Applying $P(-1, 0, 1-\varepsilon)$ for small $\varepsilon > 0$, we have

$$\begin{aligned} \frac{f(0) - f(-\varepsilon/2)}{\varepsilon/2} &\le \frac{f(1-\varepsilon)}{2-\varepsilon} \\ \implies f(0) &\le \frac{f(1-\varepsilon)}{4/\varepsilon - 2} + f(-\varepsilon/2) \le \frac{f(0)}{4/\varepsilon - 2} + f(-\varepsilon/2). \end{aligned}$$

Therefore,

$$\left(1 - \frac{1}{4/\varepsilon - 2}\right) f(0) \le f(-\varepsilon/2) \le f(0),$$

so f is continuous at 0^-, and since $1 - \frac{1}{4/\varepsilon-2} < 1$, we also have $f(0) \ge 0$.

Analogously, $P(-1+\varepsilon, 0, 1)$ gives continuity at 0^+, so f is continuous at 0. □

Claim 3 (Derivative condition). f is differentiable at every point; moreover,

$$f'(y) = \frac{f(y+a) - f(y-a)}{2a}$$

for all y, a.

Proof. Again, assume $f(-1) = f(1) = 0$. We will show f is differentiable at 0, and that $f'(0) = 0$. Limiting $\varepsilon \to 0^+$ in $P(-1, 0, 1-\varepsilon)$

$$0 \le \lim_{\varepsilon \to 0^+} \frac{f(0) - f(-\varepsilon/2)}{\varepsilon/2} \le \lim_{\varepsilon \to 0^+} \frac{f(1-\varepsilon)}{2-\varepsilon} = 0,$$

and similarly $\lim_{\varepsilon\to 0^+} \frac{f(\varepsilon/2)-f(0)}{\varepsilon/2} = 0$, so $f'(0) = 0$. $\square$

Now we will prove f quadratic; to this end, assume by shifting that $f(0) = 0$ and $f'(0) = 0$. The above claim implies that for all y, a, b,

$$\frac{f(y+a) - f(y-a)}{2a} = \frac{f(y+b) - f(y-b)}{2a}.$$

We know that for all r, $f(r) = f(-r)$. We will proceed by strong induction on n to show that $f(nr) = n^2 f(r)$ for integers $n \ge -1$. The base cases $n = -1, 0, 1$ are handled already.

To complete the inductive step, assume the hypothesis holds for integers less than n. Then take $y = n-2$, $a = 2$, $b = 1$ to derive

$$\frac{f(nr) - (n-4)^2 f(r)}{4} = \frac{(n-1)^2 f(r) - (n-3)^2 f(r)}{2}.$$

It readily follows that $f(nr) = n^2 f(r)$.

Finally, $f(p/q) = p^2/q^2 \cdot f(1)$ follows, so by continuity, $f(x) \equiv x^2 f(1)$. Since f is concave, f is either linear or a downward-facing parabola, so we are done.