USA TST 2020

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§0 Problems

Problem 1. Choose positive integers b_1, b_2, ... satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \geq r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

Problem 2. Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

Problem 3. Let $\alpha \geq 1$ be a real number. Hephaestus and Poseidon play a turn-based game on an infinite grid of unit squares. Before the game starts, Poseidon chooses a finite number of cells to be *flooded*. Hephaestus is building a *levee*, which is a subset of unit edges of the grid (called *walls*) forming a connected, non-self-intersecting path or loop.

The game begins with Hephaestus moving first. On each of Hephaestus's turns, he adds one or more walls to the levee, as long as the total length of the levee is at most αn after his nth turn. On each of Poseidon's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well.

Hephaestus wins if the levee forms a closed loop such that all flooded cells are contained in the interior of the loop — hence stopping the flood and saving the world. For which α can Hephaestus guarantee victory in a finite number of turns no matter how Poseidon chooses the initial cells to flood?

Problem 4. For a finite simplegraph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G. Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

Problem 5. Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions:

- $m > 1$ and $\gcd(m, n) = 1$;
- the numbers $P(0)$, $P^2(0)$, ..., $P^{m-1}(0)$ are not divisible by n; and
- $P^m(0)$ is divisible by n.

Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

Problem 6. Let $P_1P_2 \cdots P_{100}$ be a cyclic 100-gon, and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $P_{i-2}P_{i+1}$ and $P_{i-1}P_{i+2}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points Q_1, Q_2, ..., Q_{100} are concyclic.

§1 USA TST 2020/1 (Carl Schildkraut, Milan Haiman)

Problem 1 (USA TST 2020/1)
Choose positive integers $b_1, b_2, \dots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \frac{b_4}{4^2} > \cdots$$

and let r denote the largest real number satisfying $\frac{b_n}{n^2} \ge r$ for all positive integers n. What are the possible values of r across all possible choices of the sequence (b_n)?

The answer is $0 \le r \le 1/2$.

Claim 1. $r = 1/2$ works, and is maximal.

Proof. To achieve $r = 1/2$, take $b_n = n(n+1)/2$, from which

$$\frac{b_n}{n^2} = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n},$$

which clearly satisfies the problem condition. We inductively show that $b_n \le n(n+1)/2$. The base case has been given to us. Now, if the hypothesis holds for all integers less than n, then

$$\frac{b_n}{n^2} < \frac{b_{n-1}}{(n-1)^2} \le \frac{n}{2(n-1)} \implies b_n < \frac{n^3}{2(n-1)}.$$

It is easy to verify the largest possible b_n is $n(n+1)/2$, as claimed. $\square$

Claim 2. All $r < 1/2$ work.

Proof. Consider the sequence (a_n) defined by $a_n := \lceil kn^2 \rceil + n$. Since a_n is $O(n^2)$ and $k < 1/2$, there exists N such that for all $n \ge N$, $a_n/n^2 < 1/2$. I claim the sequence

$$b_n := \begin{cases} n(n+1)/2 & \text{for } n < N \\ a_n & \text{for } n \ge N \end{cases}$$

works. By definition of N, $b_n/n^2 > b_{n+1}/(n+1)^2$ for $n < N$, so it suffices to verify a_n/n^2 is strictly decreasing for $n \ge N$.

In other words, we want to show that

$$L := \frac{\lceil kn^2 \rceil + n}{n^2} > \frac{\lceil k(n+1)^2 \rceil + n + 1}{(n+1)^2} =: R$$

for all $n \ge N$. Since $\lceil kn^2 \rceil \ge kn^2$,

$$L \ge \frac{kn^2 + n}{n^2} = k + \frac{1}{n},$$

and similarly since $\lceil k(n+1)^2 \rceil < k(n+1)^2 + 1$,

$$R < \frac{k(n+1)^2 + n + 2}{(n+1)^2} = k + \frac{n+2}{(n+1)^2},$$

so it suffices to verify that

$$\frac{1}{n} \ge \frac{n+2}{(n+1)^2} \iff (n+1)^2 \ge n(n+2),$$

which is true. $\square$

Combining these two claims, we are done.

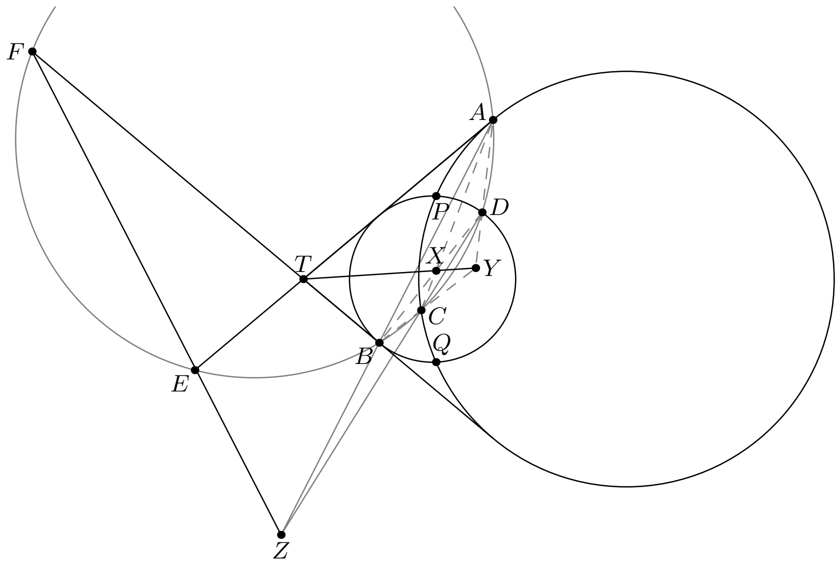
§2 USA TST 2020/2 (Merlijn Staps)

Problem 2 (USA TST 2020/2)

Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

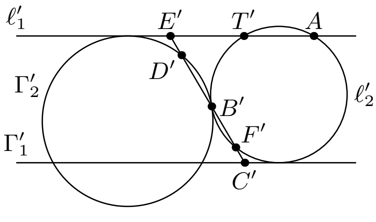
First solution, by inversion (Brandon Wang) Let ℓ_1 and ℓ_2 intersect Ω again at E and F respectively.



The key claim is this:

Claim. $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ concur.

Proof. Invert at A, using $\bullet'$ to denote the inverse, to obtain the following picture.



The homothety at B' sending Γ_2' to ℓ_2' sends ℓ_1' to Γ_1', so

$$\frac{B'C'}{B'F'} = \frac{B'E'}{B'D'} \implies B'C' \cdot B'D' = B'E' \cdot B'F',$$

whence B' lies on the radical axis of $(AC'D')$ and $(AE'F')$. Inverting back gives the desired conclusion. $\square$

Let $Z = \overline{AB} \cap \overline{CD}$, and let ℓ be the polar of Z with respect to Ω. By Brokard's theorem on $ABCD$, $\ell = \overline{XY}$, but by Brokard's theorem on $ABEF$, $\ell = \overline{TX}$. Thus T, X, Y are collinear, as desired.

Second solution, by moving points Since $ABCD$ is convex, Γ_1 and Γ_2 intersect at two points P and Q; else, the radical axis intersects all four segments AB, BC, CD, DA, which is absurd.

By radical axis theorem, X lies on $\ell := \overline{PQ}$. Animate X on ℓ. We will show that $\overline{AD}$, $\overline{BC}$, $\overline{TX}$ concur at a point Y.

Then C and D move projectively on their respective circles, so $\overline{AC}$, $\overline{BD}$ each have degree 2 and $\overline{TX}$ has degree 1. The concurrence has degree 5, so we need to verify the hypothesis for 6 values of X.

- Take X at infinity along ℓ. Then C and D are the reflections of A and B in the line through the centers of Γ_1 and Γ_2, so $Y = T$.
- Take $X = \ell \cap \overline{AB}$. Then A, B, C, D collinear, so the result is clear.
- Take $X = \ell \cap \overline{AT}$. Then $C = A$, so $Y = A$, which lies on $\overline{TX}$. The case $X = \ell \cap \overline{BT}$ follows analogously.
- Take $X = P$. Then, $C = D = P$, so $Y = P$, from which the conclusion is clear. The case $X = Q$ follows analogously.

This completes the proof.

> **Remark.** Edward Wan notes that we can instead move the center O of Ω and show that $-1 = T(AB; XZ)$, where $Z = \overline{AB} \cap \overline{CD}$. It can be shown that $O \to X$ and $O \to Z$ are projective, so this reduces the problem to three cases.

> **Remark.** I think working in $\mathbb{CP}^2$ allows us to discard the condition that Γ_1 and Γ_2 intersect (that is, $ABCD$ convex) by choosing P and Q as their non-real intersections if they do not intersect.

§3 USA TST 2020/3 (Nikolai Beluhov)

Problem 3 (USA TST 2020/3)

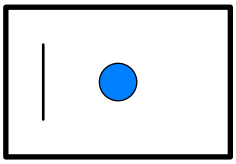
Let $\alpha \geq 1$ be a real number. Hephaestus and Poseidon play a turn-based game on an infinite grid of unit squares. Before the game starts, Poseidon chooses a finite number of cells to be *flooded*. Hephaestus is building a *levee*, which is a subset of unit edges of the grid (called *walls*) forming a connected, non-self-intersecting path or loop.

The game begins with Hephaestus moving first. On each of Hephaestus's turns, he adds one or more walls to the levee, as long as the total length of the levee is at most αn after his nth turn. On each of Poseidon's turns, every cell which is adjacent to an already flooded cell and with no wall between them becomes flooded as well.

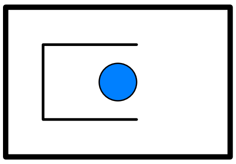
Hephaestus wins if the levee forms a closed loop such that all flooded cells are contained in the interior of the loop — hence stopping the flood and saving the world. For which α can Hephaestus guarantee victory in a finite number of turns no matter how Poseidon chooses the initial cells to flood?

The answer is $\alpha > 2$.

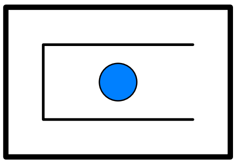
Proof of sufficiency: Take some $\alpha > 2$. We show it is possible to contain the flood. Our strategy is as follows. Here the blue circle is a large region (that grows in both directions at a rate of 1 cell per move) that contains all the flooded cells.



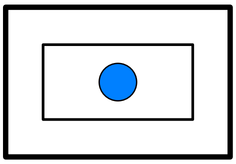
Step I: Build



Step II: Engulf



Step III: Zoom



Step IV: Eat

I. **Build a giant wall.** The total vertical height of the flood changes by at most 2 a move. Start by building a wall sufficiently far away of arbitrary height. Since $\alpha > 2$, the wall can be arbitrarily tall compared to the flood, while remaining a constant distance away from the center of the flood (since the wall can stop the flood from spreading to the other side).

II. **Engulf the flood.** After the wall is sufficiently large, begin constructing walls rightward until the rightmost point on our walls is to the right of the rightmost point of the flood.

   The flood moves rightward at a rate of at most 1 cell per move, while we can alternate between extending the top wall and the bottom wall, each increasing at a rate of $\alpha/2 > 1$ cells per move. If the original wall was large enough, the wall can extend past the flood without colliding into it, as the distance from the rightmost point of the wall and the rightmost point of the flood decreases by $\alpha/2 - 1$ cells each move.

III. **Zoom past the flood.** Now, we essentially repeat the above process. The wall can be built rightward at a rate of $\alpha/2 > 1$, so we may extend an arbitrarily large distance past the rightmost point of the flood.

IV. **Eat the flood.** Finally, build the eastern wall. If we have "zoomed" sufficiently far past the flood, we can contain the entire flood, thus completing the process.

Thus if $\alpha > 2$, Hephaestus can stop the flood and save the world.

Proof of necessity: Given two cells A and B in the flood, let L be the shortest path between A and B (including the endpoints) that does not intersect the levee, say it has length $|L|$, and furthermore say the levee has perimeter P.

Claim. For any shortest path L between two cells A and B, we have $2(|L|+1) \le P$.

Proof. From every cell in L, draw rays emanating from the center of that cell that do not lie along L; thus two rays are drawn from every cell except the endpoints, from which three are drawn. Stop these rays once they hit the levee (so that they are now segments), and call them *beams*.

I claim that no point on the levee lies on two or more beams. Beams are either parallel or perpendicular, but perpendicular beams intersect at the center of a cell, and thus not on the levee, so for two beams to intersect on the levee, the lines containing them must coincide and the corresponding rays must be in the same direction.

Let these two rays emanate from X and Y. Then by definition of these beams, L does not contain segment XY. But $\overline{XY}$ does not intersect the levee, so we may instead go directly from X to Y, contradicting the assumption that L is the shortest path.

With this, the claim readily follows. □

Suppose Poseidon starts with a flooded square A surrounded by four flooded squares, and assume Hephaestus can stop the flood. Call the final state of the levee when the flood is sealed off the *final levee*. Let B be a point in the contained flood, and let L be the shortest path between A and B that does not intersect the final levee. Suppose that $|L|$ is maximal among all B in the contained flood. Again let P be the perimeter of the final levee.

Note that if there is a wall at any point, it must be a part of the final levee. The flood will grow along L until it reaches B. Since it already has a 1 cell head-start (since A is surrounded by four flooded squares) and we have assumed that $|L|$ is maximal, at most $|L|-1$ moves have passed. It follows that Hephaestus has built at most $\alpha(|L|-1)$ walls, so

$$\alpha(|L|-1) \ge P \ge 2(|L|+1) \implies \alpha > 2,$$

and we are done.

§4 USA TST 2020/4 (Mehtaab Sawhney, Zack Chroman)

Problem 4 (USA TST 2020/4)

For a finite simplegraph G, we define G' to be the graph on the same vertex set as G, where for any two vertices $u \neq v$, the pair $\{u, v\}$ is an edge of G' if and only if u and v have a common neighbor in G. Prove that if G is a finite simple graph which is isomorphic to $(G')'$, then G is also isomorphic to G'.

Define a sequence of graphs by $G_0 = G$ and $G_n = G'_{n-1}$. I will show that for all connected graphs G, if $G \simeq G_n$ for some positive integer n, then $G \simeq G_1$. To see how this finishes the problem, note that for any graph G, if $G \simeq G''$, then the connected components of G must have been permuted. Keep permuting the connected components until they map to themselves.

Henceforth G is connected.

Claim 1. If some vertex v in G has degree ≥ 3, then for all triples of neighbors v_1, v_2, v_3 of v, all three edges of $\triangle v_1 v_2 v_3$ are in G.

Proof. Let $t(G)$ be the number of triangles in G. Assume for contradiction $\triangle v_1 v_2 v_3$ is not in G. Note that for any triangle δ in any graph G, δ is also in G'. Since $\triangle v_1 v_2 v_3$ is in G', $t(G_n) \geq t(G') > t(G)$, contradiction. □

Claim 2. If some vertex in G has degree $d \geq 3$, then G is a clique.

Proof. We proceed inductively on the number of vertices n. The base case, $n = 4$, is by Claim 1. Now assume the claim holds for all graphs with less than n vertices.

Let v be the vertex of minimal degree, and assume that G is not a clique. I claim that if we delete v, the resulting graph still contains a vertex of degree ≥ 3. Otherwise, v must be connected to a vertex w of degree 3, and all vertices not neighbors with v have degree ≤ 2, so by Claim 1, v is also connected to the other neighbors w_1 and w_2 of w, so its degree is at least 3. Unless v is connected to all vertices of G (contradicting the assumption G is not a clique), there is a vertex of degree less than v, contradiction.

By the inductive hypothesis, removing v from the graph results in a clique. Applying Claim 1 on all vertices in the clique proves that v is connected to all other vertices, so G is a clique, as claimed. □

Cliques obviously obey the problem statement, so assume all vertices have degree at most 2. Then G is either a singleton, a long chain, or a polygon; we ignore the singleton, which is easy to settle. Of these, only for a polygon with an odd number of sides is G'' connected. In this case, if G is the polygon $V_1V_2V_3 \cdots V_{n-1}V_n$ (n odd), then G' is $V_1V_3V_5 \cdots V_{n-2}V_1$, so $G \simeq G'$, and we are done.

§5 USA TST 2020/5 (Carl Schildkraut)

Problem 5 (USA TST 2020/5)
Find all integers $n \geq 2$ for which there exists an integer m and a polynomial $P(x)$ with integer coefficients satisfying the following three conditions:

- $m > 1$ and $\gcd(m, n) = 1$;
- the numbers $P(0)$, $P^2(0)$, ..., $P^{m-1}(0)$ are not divisible by n; and
- $P^m(0)$ is divisible by n.

Here P^k means P applied k times, so $P^1(0) = P(0)$, $P^2(0) = P(P(0))$, etc.

All n for which $\operatorname{rad}(n)$ is the product of the first k primes (for some k) fail, while all other n work.

Let $\mathbf{period}(P \mod n)$ denote the smallest positive integer m with $P^m(0) \equiv 0 \pmod{n}$, and infinity if such m does not exist. The key observation is

$$\mathbf{period}(P \bmod n) = \operatorname{lcm}_{p^e \| n} \mathbf{period}\left(P \bmod p^e\right), \qquad (*)$$

which follows from Chinese Remainder theorem.

Construction: We first construct prime powers:

Claim 1. Given a prime power p^e, for each $1 \leq m < p$ there is a polynomial $P \in (\mathbb{Z}/p^e\mathbb{Z})[X]$ with $\mathbf{period}(P \bmod p^e) = m$

Proof. Just take

$$P(X) = X + 1 - \frac{m}{(m-1)!} \cdot X(X-1)\cdots(X-m+2).$$

We can check $P(0) = 1$, $P(1) = 2$, ..., $P(m-2) = m-1$, $P(m-1) = 0$. □

For valid n, take $p^e \| n$ with $q < p$ a prime not dividing n, and consider the polynomial Q such that $\mathbf{period}(P \bmod p^e) = q$. Let t be a multiple of n/p^e with $t \equiv 1 \pmod{p^e}$ (which exists by Chinese Remainder theorem), and set $P = t \cdot Q$. Then P works by $(*)$.

Proof of necessity: What follows is more-or-less the converse of Claim 1. By $(*)$ it is sufficient to prove the claim.

Claim 2. Given a prime power p^e, a polynomial P, and a prime q, if $q \mid \mathbf{period}(P \bmod p)$, then $q \leq p$.

Proof. We induct on e. Clearly $\mathbf{period}(P \bmod p) \leq p$ by Pigeonhole on $P(0)$, $P^2(0)$, ..., $P^p(0)$. By the same argument, we have $\mathbf{period}(P \bmod p^e) \mid \mathbf{period}(P \bmod p^{e+1})$ and

$$\frac{\mathbf{period}(P \bmod p^{e+1})}{\mathbf{period}(P \bmod p^e)} \in \{1, 2, \ldots, p\}.$$

The inductive step follows. □

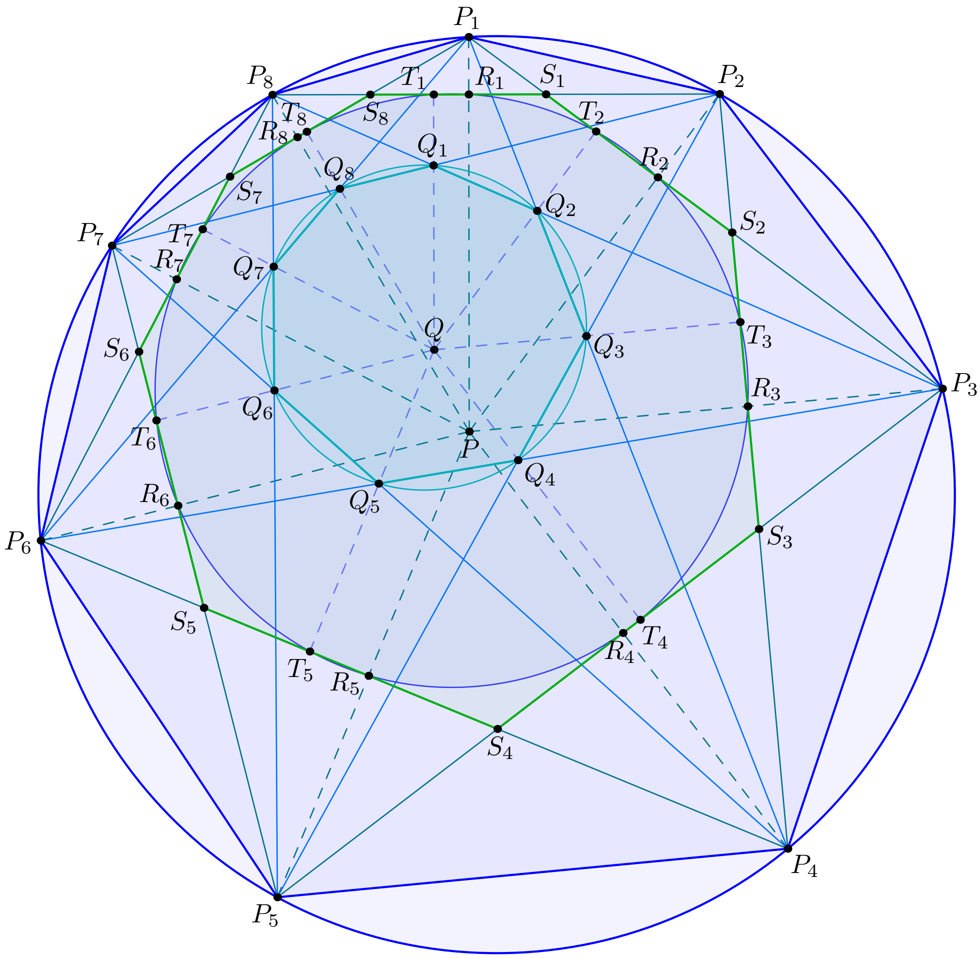
§6 USA TST 2020/6 (Michael Ren)

Problem 6 (USA TST 2020/6)

Let $P_1P_2\cdots P_{100}$ be a cyclic 100-gon, and let $P_i = P_{i+100}$ for all i. Define Q_i as the intersection of diagonals $P_{i-2}P_{i+1}$ and $P_{i-1}P_{i+2}$ for all integers i.

Suppose there exists a point P satisfying $\overline{PP_i} \perp \overline{P_{i-1}P_{i+1}}$ for all integers i. Prove that the points Q_1, Q_2, …, Q_{100} are concyclic.

Replace 100 with n. Shown below is an example for $n = 8$. Let $R_i = \overline{PP_i} \cap \overline{P_{i-1}P_{i+1}}$ be the foot from P to $\overline{P_{i-1}P_{i+1}}$, and let $S_i = \overline{P_{i-1}P_{i+1}} \cap \overline{P_iP_{i+2}}$.



Claim 1. $R_1R_2\cdots R_n$ is cyclic.

Proof. Since $\angle P_iR_iP_{i+1} = \angle P_iR_{i+1}P_{i+1} = 90^\circ$ for each i, we have $PP_i \cdot PR_i = PP_{i+1} \cdot PR_{i+1}$, meaning inversion at P with radius $\sqrt{PP_i \cdot PR_i}$, which is fixed, sends the circumcircle of $P_1P_2\cdots P_n$ to that of $R_1R_2\cdots R_{100}$. □

Claim 2. $\overline{Q_iQ_{i+1}} \parallel \overline{R_iR_{i+1}}$ for all i.

Proof. By Reim's theorem on $(P_iR_iR_{i+1}P_{i+1})$ and $(P_{i-1}P_iP_{i+1}P_{i+2})$, $\overline{Q_iQ_{i+1}} = \overline{P_{i-1}P_{i+2}} \parallel \overline{R_iR_{i+1}}$. □

Since P has a pedal circle in $S_1S_2\cdots S_{100}$, the reflection Q over the center of the pedal circle is the isogonal conjugate of P in $S_1S_2\cdots S_{100}$. Let T_i be the foot from Q to $\overline{P_{i-1}P_{i+1}}$, so that there is a fixed circle through both R_i and T_i for all i.

Claim 3. $\triangle P_{i-1}P_iP_{i+1}$ and $\triangle T_{i-1}T_iT_{i+1}$ are homothetic.

Proof. By Reim's theorem on $(P_iR_iR_{i-1}P_{i-1})$ and $(R_iT_iT_{i-1}R_{i-1})$, we have $\overline{P_iP_{i-1}} \parallel \overline{T_iT_{i-1}}$, and similarly $\overline{P_iP_{i+1}} \parallel \overline{T_iT_{i+1}}$. Also $P_iS_{i-1}\cdot P_iR_{i-1} = P_iR_i \cdot P_iP = P_iS_{i+1}\cdot P_iR_{i+1}$, so $R_{i-1}S_{i-1}S_iR_{i+1}$ is cyclic, and $\overline{P_{i-1}P_{i+1}} \parallel \overline{R_{i-1}R_{i+1}}$ by Reim's theorem on $(R_{i-1}S_{i-1}S_iR_{i+1})$ and $(R_{i-1}T_{i-1}T_{i+1}R_{i+1})$. $\square$

Claim 4. $\overline{QS_i} \perp \overline{Q_iQ_{i+1}}$ for all i.

Proof. Since S_i is the orthocenter of $\triangle PP_iP_{i+1}$, we know $\overline{PS_i} \perp \overline{P_iP_{i+1}}$. Note that $\overline{S_iP}$ and $\overline{S_iQ}$ are isogonal wrt. $\angle P_iS_iP_{i+1}$, and $\overline{P_iP_{i+1}}$ and $\overline{P_{i-1}P_{i+2}}$ are isogonal, so $\overline{QS_i} \perp \overline{P_{i-1}P_{i+2}}$, as claimed. $\square$

Claim 5. Q, Q_i, T_i are collinear for all i.

Proof. By the parallel lines from Claim 3,

$$\frac{T_iS_{i-1}}{S_{i-1}P_{i-1}} = \frac{T_iT_{i-1}}{P_iP_{i-1}} = \frac{T_iT_{i+1}}{P_iP_{i+1}} = \frac{T_iS_i}{S_iP_{i+2}}.$$

Hence a homothety at S sends $\triangle Q_iP_{i-1}P_{i+1}$ to $\triangle H_iS_{i-1}S_i$ for some point H_i with $\overline{QS_i} \perp \overline{H_iS_{i-1}}$ and $\overline{QS_{i-1}} \perp \overline{H_iS_i}$. It follows that H_i is the orthocenter of $\triangle Q_iS_{i-1}S_i$, so H_i lies on $\overline{QT_i}$. This is sufficient, since $H_i \in \overline{T_iQ_i}$ by the homothety. $\square$

From this, $\triangle PR_iR_{i+1}$ and $\triangle QQ_iQ_{i+1}$ for each i, so $R_1R_2\cdots R_n \sim Q_1Q_2\cdots Q_n$. Since the former is cyclic, so is the latter, thus concluding the proof.