

TSTST 2019

Compiled by Eric Shen

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§0 Problems

Problem 1. Find all binary operations $\diamond : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ (meaning $\diamond$ takes pairs of positive real numbers) such that for any real numbers $a, b, c > 0$,

- the equation $a \diamond (b \diamond c) = (a \diamond b) \cdot c$ holds; and
- if $a \geq 1$ then $a \diamond a \geq 1$.

Problem 2. Let ABC be an acute triangle with circumcircle Ω and orthocenter H . Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q , respectively. Denote by ω the circumcircle of $\triangle ADE$.

- Show that lines PE and QD meet on ω .
- Prove that if ω passes through H , then lines PD and QE meet on ω as well.

Problem 3. On an infinite square grid we place finitely many *cars*, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a *move*, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

Problem 4. Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we are given any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C .

Problem 5. Let ABC be an acute triangle with orthocenter H and circumcircle Γ . A line through H intersects segments AB and AC at E and F , respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ at a point D . Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ .

Problem 6. Suppose P is a polynomial with integer coefficients such that for every positive integer n , the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant?

(A *Fibonacci number* is an element of the sequence $F_0, F_1, \dots$ defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.)

Problem 7. Let $f : \mathbb{Z} \rightarrow \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y . Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x .

Problem 8. Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(\mathcal{S})$ denote the number of ways to draw 8 line segments with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

Problem 9. Let ABC be a triangle with incenter I . Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P , and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q . Prove that $IP = IQ$.

§1 TSTST 2019/1 (Evan Chen)

Problem 1

Find all binary operations $\diamond : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ (meaning $\diamond$ takes pairs of positive real numbers) such that for any real numbers $a, b, c > 0$,

- the equation $a \diamond (b \diamond c) = (a \diamond b) \cdot c$ holds; and
- if $a \geq 1$ then $a \diamond a \geq 1$.

The answer is $\times$ and $\div$. It is easy to check that these work. We now show these are the only solutions.

Claim 1. If $a \diamond p = a \diamond q$, then $p = q$.

Proof. Check that

$$(a \diamond a) \cdot p = a \diamond (a \diamond p) = a \diamond (a \diamond q) = (a \diamond a) \cdot q,$$

whence $p = q$. □

Claim 2. If $p \diamond a = q \diamond a$, then $p = q$.

Proof. Check that

$$(a \diamond p) \cdot a = a \diamond (p \diamond a) = a \diamond (q \diamond a) = (a \diamond q) \cdot a,$$

whence $a \diamond p = a \diamond q$ and $p = q$. □

Claim 3. $a \diamond 1 = a$ and $1 \diamond (1 \diamond a) = a$.

Proof. For the first part, note that $a \diamond (a \diamond 1) = (a \diamond a) \cdot 1 = a \diamond a$, so by Claim 1, $a \diamond 1 = a$, as desired. For the second part, check that $1 \diamond (1 \diamond a) = (1 \diamond 1) \cdot a = a$. □

Claim 4. $a \diamond b = a \cdot (1 \diamond b)$.

Proof. Check that

$$a \diamond b = a \diamond (1 \diamond (1 \diamond b)) = (a \diamond 1) \cdot (1 \diamond b) = a \cdot (1 \diamond b),$$

as requested. □

Now, let $f(b) = 1 \diamond b$, so that $a \diamond b = af(b)$. Our given functional equation rewrites to

$$af(bf(c)) = acf(b) \implies f(bf(c)) = cf(b).$$

However, by Claim 1, f is injective, and by Claim 4, f is an involution, so plugging in $f(c)$ as c gives $f(bc) = f(b)f(c)$. Hence, $g(x) = \ln f(e^x)$ satisfies $g(b+c) = g(b) + g(c)$. The second condition implies that $g(x) \geq -x$ for all $x \geq 0$, so g is bounded, and thus $g(x) \equiv kx$ for some k .

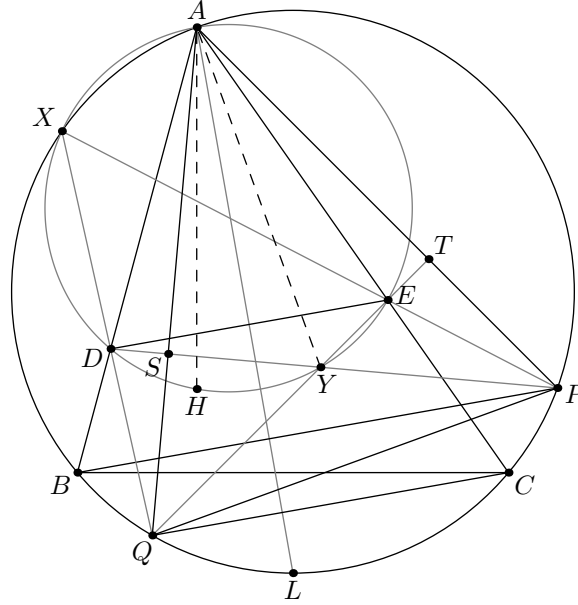
It follows that $f(x) \equiv x^k$, but substituting yields $k = \pm 1$. Hence, $a \diamond b = a \cdot b$ or $a \div b$, as desired.

§2 TSTST 2019/2 (Merlijn Staps)

Problem 2

Let ABC be an acute triangle with circumcircle Ω and orthocenter H . Points D and E lie on segments AB and AC respectively, such that $AD = AE$. The lines through B and C parallel to $\overline{DE}$ intersect Ω again at P and Q , respectively. Denote by ω the circumcircle of $\triangle ADE$.

- (a) Show that lines PE and QD meet on ω .
- (b) Prove that if ω passes through H , then lines PD and QE meet on ω as well.



Solution to part (a) Let $X = \overline{PE} \cap \overline{QD}$. It is easy to see that $\angle APQ = \angle ACQ = \angle AED$ and $\angle PQA = \angle PBA = \angle EDA$, whence $\triangle ADE \sim \triangle APQ$, and A is the Miquel point of $DEPQ$.

Hence, X lies on both Ω and ω , as desired.

First solution to part (b), using reverse construction Let Y be the orthocenter of $\triangle APQ$ and set $D = \overline{AB} \cap \overline{PY}$ and $E = \overline{AC} \cap \overline{QY}$. Also let $S = \overline{AQ} \cap \overline{PY}$ and $T = \overline{AP} \cap \overline{QY}$.

Claim 1. $AD = AE$ and Y lies on ω .

Proof. Since $\widehat{BQ} = \widehat{CP}$, $\triangle ADS \cong \triangle AET$. It is immediate that $AD = AE$, and furthermore

$$\angle DYE = \angle SYT = \angle SAT = \angle QAP = \angle DAE,$$

as desired. □

Claim 2. H lies on ω .

Proof. Let O be the center of Ω . Clearly $AP = AQ$, $OP = OQ$, and $YP = YQ$, so A, O, P lie on the perpendicular bisector of $\overline{PQ}$.

Now, if R denotes the radius of Ω , $AH = 2R \cos \angle BAC = 2R \cos \angle QAP = AY$, and furthermore $\overline{AH}$ and $\overline{AY}$ are isogonal, so H lies on ω , as desired. □

Since D and E are unique, we are done.

Second solution to part (b), using complex numbers (Michael Ma) Let L be the midpoint of arc $\widehat{BC}$ not containing A . Toss on the complex plane, with lowercase letters denoting complex numbers; set $a = u^2$, $b = v^2$, $c = w^2$, and $l = -vw$. We first prove a crucial claim:

Claim. $\overline{DH} \parallel \overline{BL}$ and $\overline{EH} \parallel \overline{CL}$.

Proof. This is just angle chasing.

$$\angle ADH = 180^\circ - \angle DHA - \angle HAD = 90^\circ - \angle DEA + \angle B = \frac{1}{2}\angle A + \angle B = \angle ABL,$$

as desired. □

Now, everything is computable. Since $\overline{AM} \perp \overline{BP}$, we have that

$$am + bp = 0 \implies p = -\frac{am}{b} = \frac{u^2w}{v},$$

and similarly $q = \frac{u^2v}{w}$.

To compute d , note that we have $D \in \overline{AB}$, so $\bar{d} = \frac{u^2+v^2-d}{u^2v^2}$. Furthermore, $\overline{DH} \parallel \overline{BL}$, so

$$\frac{d-h}{b-m} \in \mathbb{R} \implies \frac{d-u^2-v^2-w^2}{v^2+vw} \in \mathbb{R}.$$

Now, we have that

$$\begin{aligned} \frac{d-u^2-v^2-w^2}{v^2+vw} &= \frac{\frac{u^2+v^2-d}{u^2v^2} - \frac{1}{u^2} - \frac{1}{v^2} - \frac{1}{w^2}}{\frac{1}{v^2} + \frac{1}{vw}} \\ \implies d-u^2-v^2-w^2 &= v^3w \left(-\frac{d}{u^2v^2} - \frac{1}{w^2} \right) \\ \implies d &= \frac{u^2(u^2w + v^2w + w^3 - v^3)}{w(u^2 + vw)}. \end{aligned}$$

Similarly,

$$e = \frac{u^2(u^2v + vw^2 + v^3 - w^3)}{v(u^2 + vw)}.$$

Notice that

$$d-p = \frac{(w-v)(v^3-u^2w)u^2}{vw(u^2+vw)} \implies -\frac{d-p}{e-q} = \frac{v^2-u^2w}{w^2-u^2v}.$$

Finally, we want

$$x = -\frac{b-a}{c-a} \bigg/ \frac{d-p}{e-q} = \frac{(v^2-u^2)(w^3-u^2v)}{(w^2-u^2)(v^3-u^2w)}$$

to be real, but

$$\bar{x} = \left(\frac{u^2-v^2}{u^2v^2} \cdot \frac{u^2v-w^3}{u^2vw^3} \right) \bigg/ \left(\frac{u^2-w^2}{u^2w^2} \cdot \frac{u^2w-v^3}{u^2v^3w} \right) = x,$$

and we are done.

Third solution to part (b), using orthocenter reflections Let H_B and H_C denote the reflections of H across $\overline{CA}$ and $\overline{AB}$ respectively.

Claim 1. H_C lies on $\overline{PD}$ and H_B lies on $\overline{QE}$.

Proof. Since $\triangle ADE \sim \triangle APQ$,

$$\angle AH_C D = \angle DHA = \angle DEA = \angle AQP = \angle AH_C P,$$

as requested. □

Claim 2. Let Y denote the reflection of H across the A -angle bisector. Y lies on ω and also on $\overline{PD}$ and $\overline{QE}$.

Proof. Clearly Y lies on ω , as $DEYH$ is an isosceles trapezoid. Check that

$$\angle YDE = \angle YAE = \angle DAH = \angle H_C AB = \angle H_C PB = \angle PDE,$$

as desired. □

Hence, $Y = \overline{PD} \cap \overline{QE}$ lies on ω , and we are done.

§3 TSTST 2019/3 (Nikolai Beluhov)

Problem 3

On an infinite square grid we place finitely many *cars*, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that the cell immediately in front of each car is empty, and moreover no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.). In a *move*, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times.

Call a car *horizontal* if it is left-facing or right-facing, and *vertical* otherwise. Color the columns black and white in an alternating manner.

Surround all the cars in a box. It is sufficient to show that the cars can all leave the box. Perform the following procedure:

- Move all horizontal cars in a black column to a white column. This is possible because the cell immediately in front of each car is empty.
- Clear the black columns of vertical cars. This is possible because all cars in a black column must face the same direction.
- Move all the horizontal cars into black columns. This is possible because all vertical cars in black columns have left.
- Clear the white columns of vertical cars, which is possible for similar reasons as above.
- Clear the box of horizontal cars, which must all face the same direction

With the box sufficiently big, all cars can move infinitely many times, as desired.

§4 TSTST 2019/4 (Merlijn Staps)

Problem 4

Consider coins with positive real denominations not exceeding 1. Find the smallest $C > 0$ such that the following holds: if we are given any 100 such coins with total value 50, then we can always split them into two stacks of 50 coins each such that the absolute difference between the total values of the two stacks is at most C .

The answer is $\frac{50}{51}$. This is possible by giving 49 coins a value of 1 and 51 coins a value of $\frac{1}{51}$. Now, suppose the values of the coins are denoted $a_1 \leq a_2 \leq \dots \leq a_{100}$. Call a gap between two consecutive elements a_n and a_{n+1} *big* if $a_{n+1} - a_n > \frac{50}{51}$.

Claim. A gap can be big only if $n = 50$.

Proof. Assume that $k < 50$ is a big gap. Then, $a_i < \frac{1}{51}$ for all $i \leq k$ and $a_i > \frac{50}{51}$ for all $i > k$. Hence,

$$\sum_{i=1}^{100} a_i > \frac{50}{51}(100 - k) \geq \frac{50 \cdot 51}{51} = 50,$$

a contradiction. Similarly, if $k > 50$ is a big gap, then

$$\sum_{i=1}^{100} a_i < \frac{1}{51}k + (100 - k) = 100 - \frac{50}{51}k \geq 100 - \frac{50 \cdot 51}{51} = 50,$$

the desired contradiction. □

It is immediate that $a_{50} \leq \frac{50}{51}$ and $a_{51} \geq \frac{1}{51}$. Consider

$$\begin{aligned} L &= (a_1 + a_3 + \dots + a_{49}) + (a_{52} + a_{54} + \dots + a_{100}), \\ R &= (a_2 + a_4 + \dots + a_{50}) + (a_{51} + a_{53} + \dots + a_{99}). \end{aligned}$$

Compute that

$$\begin{aligned} L - R &= a_1 + (a_3 - a_2) + (a_5 - a_4) + \dots + (a_{49} - a_{48}) \\ &\quad - a_{50} + (a_{52} - a_{51}) + (a_{54} - a_{53}) + \dots + (a_{100} - a_{99}) \\ &\geq a_1 - a_{50} \geq -\frac{50}{51}, \end{aligned}$$

and

$$\begin{aligned} R - L &= (a_2 - a_1) + (a_4 - a_3) + \dots + (a_{50} - a_{49}) + a_{51} \\ &\quad + (a_{53} - a_{52}) + (a_{55} - a_{54}) + \dots + (a_{99} - a_{98}) - a_{100} \\ &\geq a_{51} - a_{100} \geq -\frac{50}{51}, \end{aligned}$$

whence $|L - R| \leq \frac{50}{51}$, and we are done.

§5 TSTST 2019/5 (Gunmay Handa)

Problem 5

Let ABC be an acute triangle with orthocenter H and circumcircle Γ . A line through H intersects segments AB and AC at E and F , respectively. Let K be the circumcenter of $\triangle AEF$, and suppose line AK intersects Γ at a point D . Prove that line HK and the line through D perpendicular to $\overline{BC}$ meet on Γ .

First solution, by radical axes Let T lie on Γ with $\overline{DT} \perp \overline{BC}$, and let line TH intersect Γ again at a point $S \neq T$.

Claim. Quadrilaterals $BEHS$ and $CFHS$ are cyclic.

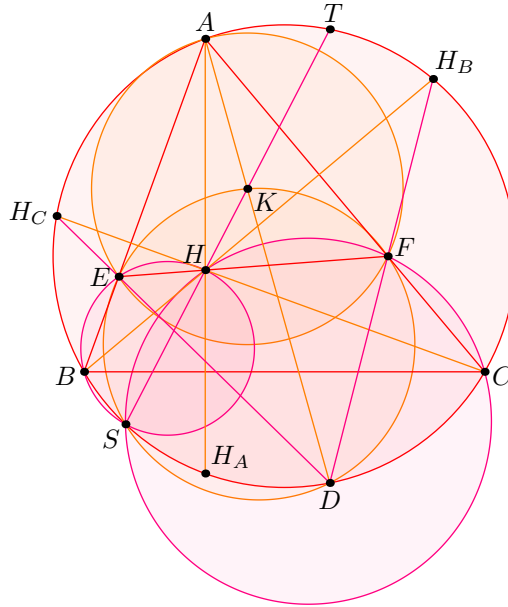
Proof. This is just angle chasing:

$$\angle BSH = \angle BDT = 90^\circ - \angle CBD = 90^\circ - \angle FAK = \angle AEF = \angle BEH,$$

as requested. $\square$

Now, $\angle KEF = 90^\circ - \angle FAE = \angle EBH$, so $\overline{KE}$ and $\overline{KF}$ are tangent to $(BEHS)$ and $(CFHS)$, respectively. However, $KE^2 = KF^2$, so K lies on line HS , and the desired conclusion follows.

Second solution, by anti-Steiner points Denote by H_A, H_B, H_C the reflections of H over $\overline{BC}, \overline{CA}, \overline{AB}$, respectively.



Claim 1. D is the anti-Steiner point of $\overline{EHF}$, and $DEKF$ is cyclic.

Proof. Let $D' = \overline{EH_C} \cap \overline{FH_B}$ be the anti-Steiner point of $\overline{EHF}$. First note that

$$\begin{aligned} \angle KED' &= \angle KEA + \angle AED' = 90^\circ - \angle AFE + \angle AEH_C \\ &= 90^\circ + \angle EFA + \angle HEA = 90^\circ + \angle EFA + \angle FEA, \end{aligned}$$

which is symmetric, whence $\angle KED' = 90^\circ + \angle EFA + \angle FEA = \angle KFD'$, and $D'EKF$ is cyclic. Now,

$$\angle AKE = 2\angle AFE = 2\angle AFH = \angle H_B FH = \angle D'FE = \angle D'KE,$$

so $D' \in \overline{AK}$, as required. $\square$

Claim 2. Let S be the Miquel point of $EFH_B H_C$; S lies on line HK .

Proof. Check that

$$\frac{SE}{SF} = \frac{EH_C}{FH_B} = \frac{EH}{FH},$$

whence $\overline{SH}$ bisects $\angle ESF$ and $\overline{SH}$ passes through K , the midpoint of arc EF on $(DEKF)$. $\square$

Claim 3. Quadrilaterals $BEHS$ and $CFHS$ are cyclic.

Proof. Notice that

$$\angle ESH = \angle ESK = \angle EFK = 90^\circ - \angle FAE = \angle ABH = \angle EBH,$$

so we are done by symmetry. $\square$

Finally, let line HK meet Γ at $T \neq S$. Note that

$$\angle ATH = \angle ATS = \angle ABS = \angle EBS = \angle EHS = \angle FHT,$$

so $\overline{AT} \parallel \overline{EF}$. However, $\overline{DH_A}$ is the reflection of $\overline{EF}$ over $\overline{BC}$, so $\angle TAH_A = \angle EHA = \angle AH_A D$, and $AD = TH_A$. Thus, $ATDH_A$ is an isosceles trapezoid, and $\overline{AH_A} \parallel \overline{DT}$. This completes the proof.

§6 TSTST 2019/6 (Nikolai Beluhov)

Problem 6

Suppose P is a polynomial with integer coefficients such that for every positive integer n , the sum of the decimal digits of $|P(n)|$ is not a Fibonacci number. Must P be constant?

(A *Fibonacci number* is an element of the sequence $F_0, F_1, \dots$ defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$.)

The answer is yes.

Lemma 1

Fibonacci numbers are surjective modulo 9.

Proof. Omitted. □

Lemma 2

Let n be the degree of P . There exists a (cubic) polynomial Q such that $P(Q(x))$ has x^{3n-1} as its only negative coefficient.

Proof. The key is that $Q(x) = x^3 + x + 1$ *almost* works for $P(x) = x^n$, except the x^{3n-1} coefficient is 0. Take large positive constants C and D , and consider

$$Q(x) = C(Dx^3 - x^2 + Dx + D).$$

With C sufficiently big, only the x^n term of P matters, so Q works. □

Now, for all sufficiently large k , consider $P(Q(10^k))$. The coefficients will all be isolated, besides the x^{3n-1} term. As we increase k , we increase the decimal digit sum of $|P(Q(10^k))|$ by 9, so there are a fixed b and c such that all numbers of the form $9a + b$ with $a \geq c$ are achievable.

Since Fibonacci numbers are periodic, we are done by Lemma 1.

§7 TSTST 2019/7 (Ankan Bhattacharya)

Problem 7

Let $f : \mathbb{Z} \rightarrow \{1, 2, \dots, 10^{100}\}$ be a function satisfying

$$\gcd(f(x), f(y)) = \gcd(f(x), x - y)$$

for all integers x and y . Show that there exist positive integers m and n such that $f(x) = \gcd(m + x, n)$ for all integers x .

Let $p \leq 10^{100}$ be a prime. Set x such that $\nu_p(f(x))$ is maximal, so that for all y ,

$$\nu_p(f(y)) = \min[\nu_p(f(x)), \nu_p(x - y)].$$

It is immediate that all y of the form $x + kp^{\nu_p(f(x))}$ have maximal $\nu_p(f(y))$, so $\nu_p(f(y))$ is maximal when

$$y \equiv x \pmod{p^{\nu_p(f(x))}}. \quad (*)$$

By the Chinese Remainder Theorem, there exists negative z obeying $(*)$, so taking $m = -z$ and $n = f(z)$ yields $f(y) \equiv \gcd(n, y + m)$, as desired.

§8 TSTST 2019/8 (Ankan Bhattacharya)

Problem 8

Let $\mathcal{S}$ be a set of 16 points in the plane, no three collinear. Let $\chi(\mathcal{S})$ denote the number of ways to draw 8 line segments with endpoints in $\mathcal{S}$, such that no two drawn segments intersect, even at endpoints. Find the smallest possible value of $\chi(\mathcal{S})$ across all such $\mathcal{S}$.

The answer is 1430. Let C_n denote the minimum possible value of $\chi(\mathcal{S})$ when there are $2n$ points.

Claim. C_n and $\chi(\mathcal{S})$ are related as follows:

- (i) $\chi(\mathcal{S}) \geq C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$.
- (ii) $\chi(\mathcal{S}) = C_n$ when $\mathcal{S}$ is a convex $2n$ -gon.

Proof. Let A be a point on the convex hull of $\mathcal{S}$, and let B be the other endpoint of the segment incident to A . Suppose that line AB splits the plane into two half-planes containing k and $n-1-k$ points respectively.

If k is odd, then there are clearly at least $C_k C_{n-1-k}$ ways to draw the segments, so

$$\chi(\mathcal{S}) \geq \sum_{k=0}^{n-1} C_k C_{n-1-k},$$

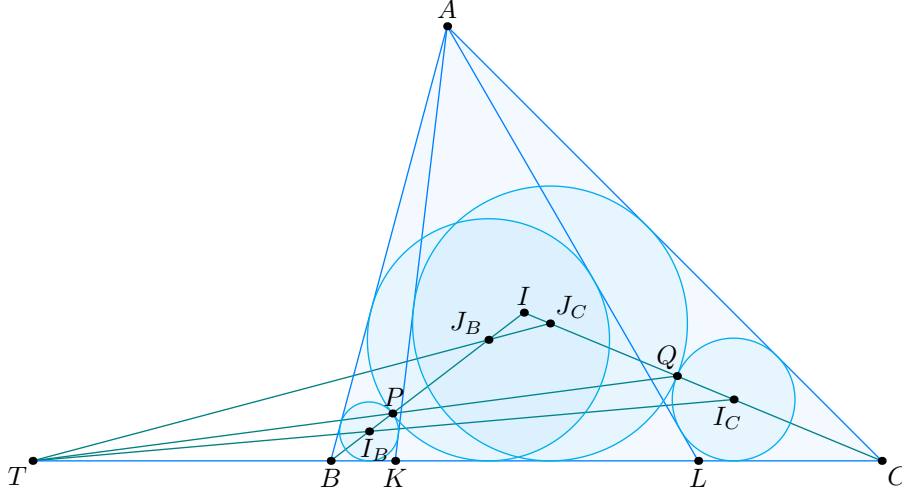
as requested. Equality occurs when $\mathcal{S}$ is a convex $2n$ -gon, as nothing can cross segment AB and $\overline{AB}$ splits $\mathcal{S}$ into two convex sets. $\square$

Since $C_0 = C_1 = 1$, C_n is the n^{th} Catalan number, so $C_8 = \frac{1}{9} \binom{16}{8} = 1430$, the answer.

§9 TSTST 2019/9 (Ankan Bhattacharya)

Problem 9

Let ABC be a triangle with incenter I . Points K and L are chosen on segment BC such that the incircles of $\triangle ABK$ and $\triangle ABL$ are tangent at P , and the incircles of $\triangle ACK$ and $\triangle ACL$ are tangent at Q . Prove that $IP = IQ$.



Let I_B, J_B, I_C , and J_C denote the incenters of $\triangle ABK, \triangle ABL, \triangle ACL$, and $\triangle ACK$, respectively. Also let $r_{\triangle XYZ}$ denote the inradius of $\triangle XYZ$ for all X, Y, Z .

Lemma

For any points K and L on $\overline{BC}$ of $\triangle ABC$, if I_B, J_B, I_C , and J_C denote the incenters of $\triangle ABK, \triangle ABL, \triangle ACL$, and $\triangle ACK$, respectively, then $\overline{BI_BI_C}, \overline{J_BJ_C}$, and $\overline{BC}$ concur at a point T .

Proof. Rotation by $\frac{1}{2}\angle A$ about A gives

$$A(BI_B; IJ_B) = A(IJ_C; CI_C) = A(CI_C; IJ_C),$$

and thus $(BI_B; IJ_B) = (CI_C; IJ_C)$ and the result follows.¹ $\square$

Considering the homothety centered at B sending (I_B) to (J_B) , we can check that the scale factor is

$$\frac{PJ_B}{PI_B} = \frac{r_{\triangle ABL}}{r_{\triangle ABK}} = \frac{BJ_B}{BI_B} \implies -1 = (BP; I_BJ_B),$$

and similarly $-1 = (CP; I_CJ_C)$. It is immediate that $\overline{PQ}$ also passes through T . By Menelaus on $\triangle I I_B I_C$,

$$-1 = \frac{IP}{PI_B} \cdot \frac{I_B T}{T I_C} \cdot \frac{I_C Q}{Q I} = \frac{IP}{Q I} \cdot \frac{I_B T}{T I_C} \cdot \frac{I_C Q}{P I_B} = \frac{IP}{Q I} \cdot \frac{r_{\triangle ABK}}{r_{\triangle ACL}} \cdot \frac{r_{\triangle ACL}}{r_{\triangle ABK}} = \frac{IP}{Q I},$$

whence $IP = IQ$, as desired.

¹Alternatively we can directly compute that

$$A(BI_B; IJ_B) = \frac{\sin \frac{1}{2}\angle BAC \cdot \sin \frac{1}{2}\angle KAL}{\sin \frac{1}{2}\angle KAC \cdot \sin \frac{1}{2}\angle BAL} = A(CI_C; IJ_C).$$