2019 Mock AIME
Tiebreaker Round Solutions

by TheUltimate123 and nukelauncher

§1 Answers

Tiebreaker Round A consisted of problems PQRST and was given to contestants who tied and scored at most 5. Tiebreaker Round B consisted of problems TUVWX and was given to contestants who tied and scored between 6 and 10 inclusive. Tiebreaker Round C consisted of problems VWXYZ and was given to contestants who scored at least 11.

Problem	Author	Answer
P	nukelauncher	016
Q	TheUltimate123	245
R	tigerche	679
S	TheUltimate123	426
T	nukelauncher	039
U	nukelauncher	033
V	TheUltimate123	692
W	TheUltimate123	376
X	TheUltimate123	130
Y	nukelauncher	919
Z	TheUltimate123	449

§2 Problems

P. Given that $n+2$ and $2n-3$ are the squares of two consecutive integers, what is the sum of all possible values of n?

Q. Suppose that x and y are real numbers such that

$$\begin{aligned} \log_3(x+y^4) &= \log_3(x-y) + \log_3(x+y), \text{ and} \\ 10 &= \log_3(x-2y) + \log_3(x+2y). \end{aligned}$$

Find x.

R. In triangle ABC, $AB=5$, $BC=8$, and $CA=7$. Let the internal angle bisector of $\angle BAC$ intersect $\overline{BC}$ at P, and let Q be the point on $\overline{AB}$ distinct from A such that $CQ=7$. The square of the area of quadrilateral $ACPQ$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

S. The positive integer N can be written as $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ in base 7, and $\underline{r}\,\underline{s}\,\underline{t}$ in base 9. Suppose that the value of the digit d is the sum of the values of a, b, and c, and that the value of the digit r is the sum of the values of s and t. Furthermore, the values of c and d are double the values of s and t, respectively. Find N.

T. Aaron and Erin are working together on a 30-question test that consists of 12 physics problems and 18 math problems. Aaron picks two distinct problems at random so that each problem has an equal chance of being chosen, and correctly solves both of them. Erin then chooses one of the 30 problems to solve so that each problem has an equal chance of being chosen. The probability that the problem Erin chose is an unsolved math problem can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

U. A sequence $\{a_i\}$ satisfies $a_{n+1} = \frac{a_n - 7}{a_n + 2}$ for all positive integers n. If $a_{2019} = 2019$, then a_1 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

V. Call a positive integer *palatable* if when expressed in binary, each contiguous block of zeros that is not a subsequence of another contiguous block of zeros has even length, and each contiguous block of ones that is not a subsequence of another contiguous block of ones has odd length. For example, $57 = 111001_2$ is palatable while $69 = 1000101_2$ is not. Find the number of palatable positive integers N such that $2^{18} < N < 2^{19}$.

W. Consider all positive integers a, b such that $\operatorname{lcm}(a^2 - 1, b^2 - 1) = 29^2 - 1$. Find the sum of all distinct values of $a + b$.

X. In triangle ABC, $AB = 26$, $BC = 42$, and $CA = 40$. Let ω be the incircle of $\triangle ABC$, and let ω_A be the circle tangent to segment BC and the extensions of lines AB and AC past B and C, respectively. Suppose that ω and ω_A are tangent to $\overline{BC}$ at P and Q, respectively, and that X and Y lie on ω and ω_A, respectively, such that $\angle AXP = \angle AYQ = 90^\circ$. If M is the midpoint of $\overline{BC}$ and Z is the intersection of $\overline{PX}$ and $\overline{QY}$, find MZ^2.

Y. Suppose that S denotes the set of all positive integers that are divisible by no primes other than 2, 3, and 7, and that $\varphi(n)$ denotes the number of positive integers not exceeding n that are relatively prime to n. Then, there exist relatively prime integers p and q such that

$$\sum_{n \in S} \left(\frac{1}{n^3} \sum_{d \mid n} d\varphi(d) \right) = \frac{p}{q}.$$

Find $p + q$.

Z. In triangle ABC, $AB = 14$, $BC = 17$, and $CA = 15$. The incircle of $\triangle ABC$ touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively; and $\overline{AD}$ meets the incircle of $\triangle ABC$ again at T. Suppose that points M and U lie on $\overline{AC}$ and points N and V lie on $\overline{AB}$ such that $\overline{MN}$ is tangent to the incircle of $\triangle ABC$ at T, and $\overline{EV}$ and $\overline{FU}$ intersect at T. Then, there exist relatively prime positive integers p and q such that $\frac{MU}{NV} = \frac{p}{q}$. Find $p + q$.

§3 Solutions

The solutions to these problems begin on the next page.

Problem P

Given that $n+2$ and $2n-3$ are the squares of two consecutive integers, what is the sum of all possible values of n?

Answer. 016

Suppose that $n+2=k^2$. Then, $2n-3=(k+1)^2$. This substitution works because if $n+2>2n-3$, then we can simply take k negative. Thus, $n=k^2-2$ and

$$(k+1)^2=2(k^2-2)-3=2k^2-7 \implies 0=k^2-2k-8=(k-4)(k+2).$$

It follows that $k=4$ or -2, whence $n=14$ and 2, respectively. The requested sum is $14+2=16$.

Problem Q

Suppose that x and y are real numbers such that

$$\log_3(x+y^4) = \log_3(x-y) + \log_3(x+y), \text{ and}$$
$$10 = \log_3(x-2y) + \log_3(x+2y).$$

Find x.

Answer. 245

The first relation gives us $x + y^4 = (x-y)(x+y) = x^2 - y^2$, so $x^2 - x = y^4 + y^2$ and

$$\left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4} = y^4 + y^2 + \frac{1}{4} = \left(y^2 + \frac{1}{2}\right)^2.$$

This implies that either $x = y^2 + 1$ or $x = -y^2$. In the latter case, the second condition gives a contradiction, so $x = y^2 + 1$. The second condition now gives

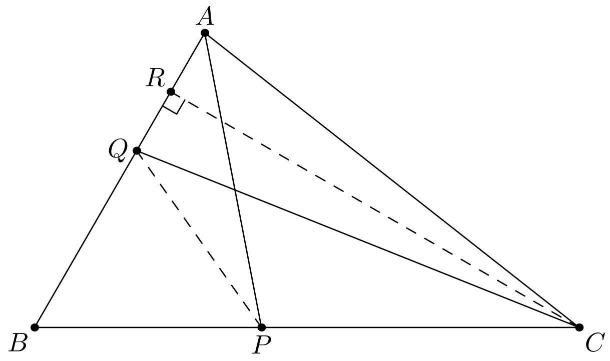
$$3^{10} = (x-2y)(x+2y) = x^2 - 4y^2 = (y^2+1)^2 - 4y^2 = (y^2-1)^2,$$

so $y^2 - 1 = 243$, and $x = y^2 + 1 = 245$, the answer.

Problem R

In triangle ABC, $AB = 5$, $BC = 8$, and $CA = 7$. Let the internal angle bisector of $\angle BAC$ intersect $\overline{BC}$ at P, and let Q be the point on $\overline{AB}$ distinct from A such that $CQ = 7$. The square of the area of quadrilateral $ACPQ$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer. 679



Check that

$$\cos B = \frac{5^2 + 8^2 - 7^2}{2 \cdot 5 \cdot 8} = \frac{1}{2},$$

so $\angle B = 60^\circ$. Then,

$$[ABC] = \frac{5 \cdot 8 \cdot \sin A}{2} = 10\sqrt{3}.$$

By the Angle Bisector Theorem,

$$\frac{BP}{BC} = \frac{AB}{AB + AC} = \frac{5}{12} \implies BP = \frac{10}{3}.$$

Now, let R be the foot of the altitude from C to $\overline{AB}$. By HL, $\triangle CAR \cong \triangle CQR$. However, since $\angle B = 60^\circ$, $BR = 4$, so $QR = AR = 1$. Hence, $BQ = 3$, and

$$[BPQ] = \frac{10/3 \cdot 3 \cdot \sin A}{2} = \frac{5\sqrt{3}}{2},$$

so

$$[ACPQ] = 10\sqrt{3} - \frac{5\sqrt{3}}{2} = \frac{15\sqrt{3}}{2} \implies [ACPQ]^2 = \frac{675}{4},$$

and the requested sum is $675 + 4 = 679$.

Problem S

The positive integer N can be written as $\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}$ in base 7, and $\underline{r}\ \underline{s}\ \underline{t}$ in base 9. Suppose that the value of the digit d is the sum of the values of a, b, and c, and that the value of the digit r is the sum of the values of s and t. Furthermore, the values of c and d are double the values of s and t, respectively. Find N.

Answer. 426

We know that

$$N = a \cdot 7^3 + b \cdot 7^2 + (2s) \cdot 7 + (a + b + 2s) = 344a + 50b + 16s.$$

Furthermore,

$$N = \left(\frac{a+b}{2} + 2s\right) \cdot 9^2 + s \cdot 9 + \left(\frac{a+b}{2} + s\right) = 41a + 41b + 172s.$$

Setting the two expressions equal, we have that $101a + 3b = 52s$. However, since s is a digit in base 7, we have that $s \in \{0, 1, 2, 3\}$.

- If $s = 0$, then $a = b = 0$ and $N = 0$, which is not positive.
- If $s = 1$, then $101a + 3b = 52$, which is impossible if $a, b < 7$.
- If $s = 2$, then $101a + 3b = 104$, so $a = b = 1$. This yields $N = 426$
- If $s = 3$, then $101a + 3b = 156$, which is impossible if $a, b < 7$.

Hence, the answer is 426.

Problem T

Aaron and Erin are working together on a 30-question test that consists of 12 physics problems and 18 math problems. Aaron picks two distinct problems at random so that each problem has an equal chance of being chosen, and correctly solves both of them. Erin then chooses one of the 30 problems to solve so that each problem has an equal chance of being chosen. The probability that the problem Erin chose is an unsolved math problem can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer. 039

By Linearity of Expectation, the expected number of physics problems Aaron solves is $2 \cdot 2/5 = 4/5$ and the expected number of math problems he solves is $2 \cdot 3/5 = 6/5$. Hence, an expected number of $56/5$ physics problems and $84/5$ math problems remain.

Now, the probability Erin chooses an unsolved math problem is simply $84/5 \div 30 = 14/25$, and the requested sum is $14 + 25 = 39$.

Problem U

A sequence $\{a_i\}$ satisfies

$$a_{n+1} = \frac{a_n - 7}{a_n + 2}$$

for all positive integers n. If $a_{2019} = 2019$, then a_1 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

Answer. 033

Let $a_1 = x$. Compute that

$$\begin{aligned}
a_2 &= \frac{x-7}{x+2}. \\
a_3 &= \frac{\frac{x-7}{x+2} - 7}{\frac{x-7}{x+2} + 2} = \frac{(x-7) - 7(x+2)}{(x-7) + 2(x+2)} = \frac{-2x-7}{x-1}. \\
a_4 &= \frac{\frac{-2x-7}{x-1} - 7}{\frac{-2x-7}{x-1} + 2} = \frac{(-2x-7) - 7(x-1)}{(-2x-7) + 2(x-1)} = x.
\end{aligned}$$

Hence, $\{a_i\}$ repeats in a period of 3, so $a_{2019} = a_3$. Then,

$$2019 = \frac{-2x-7}{x-1} \implies x = \frac{2012}{2021},$$

whence the requested remainder is $2012 + 2021 \equiv 33 \pmod{1000}$.

Problem V

Call a positive integer *palatable* if when expressed in binary, each contiguous block of zeros that is not a subsequence of another contiguous block of zeros has even length, and each contiguous block of ones that is not a subsequence of another contiguous block of ones has odd length. For example, $57 = 111001_2$ is palatable while $69 = 1000101_2$ is not. Find the number of palatable positive integers N such that $2^{18} < N < 2^{19}$.

Answer. 692

Let a_k be the number of palatable integers with k digits in base 2 that end in 0, and b_k the number that end in 1. If the number T ends in 0 in binary, we can append two 0's or one 1 to form another palatable integer, while if T ends in 1 in binary, we can append two 0's or two 1's to form another. Hence, $a_k = a_{k-2} + b_{k-2}$ and $b_k = a_{k-1} + b_{k-2}$. It is easy to check that $a_1 = 0$, $a_2 = 0$, $b_1 = 1$, and $b_2 = 0$. Then,

k	a_k	b_k
1	0	1
2	0	0
3	1	1
4	0	1
5	2	1
6	1	3
7	3	2
8	4	6
9	5	6
10	10	11
11	11	16
12	21	22
13	27	37
14	43	49
15	64	80
16	92	113
17	144	172
18	205	257
19	316	377

Since 2^{18} is palatable, the answer is $a_{19} + b_{19} - 1 = 692$.

Problem W

Consider all positive integers a, b such that $\operatorname{lcm}(a^2-1, b^2-1) = 29^2-1$. Find the sum of all distinct values of $a+b$.

Answer. 376

First assume that $a, b < 29$. Clearly $a, b \neq 1$. Note that $29^2 - 1 = 28 \cdot 30 = 2^3 \cdot 3 \cdot 5 \cdot 7$. Hence, both a and b must be solutions to $(c-1)(c+1) \mid 2^3 \cdot 3 \cdot 5 \cdot 7$. This implies that $c \not\equiv 1, 7 \pmod 8$.

Now consider powers of 7. One of a^2-1 and b^2-1 must contain a factor of 7; WLOG let it be a. This implies that $a \in \{6, 8, 13, 15, 20, 22, 27\}$. However, we can eliminate $8, 15, 20, 22, 27$ by checking divisors. Hence, $a \in \{6, 13\}$. We will take cases:

- *Case 1:* $a = 6$. Then, $a^2 - 1 = 5 \cdot 7$, so b^2-1 must be divisible by $2^3 \cdot 3$. This implies that b is odd but not 1 nor 7 mod 8, and $3 \nmid b$. Check that $b \in \{5, 11, 13, 19, 27\}$ satisfy these. It is now easy to check that only $5, 11, 13$ work.
- *Case 2:* $a = 13$. Then, $a^2-1 = 2^3 \cdot 3 \cdot 7$, so we only require b^2-1 to be divisible by 5. This implies $b \in \{4, 6, 9, 11, 14, 16, 19, 21, 24, 26\}$. However, it is easy to check that only $4, 6, 11$ work.

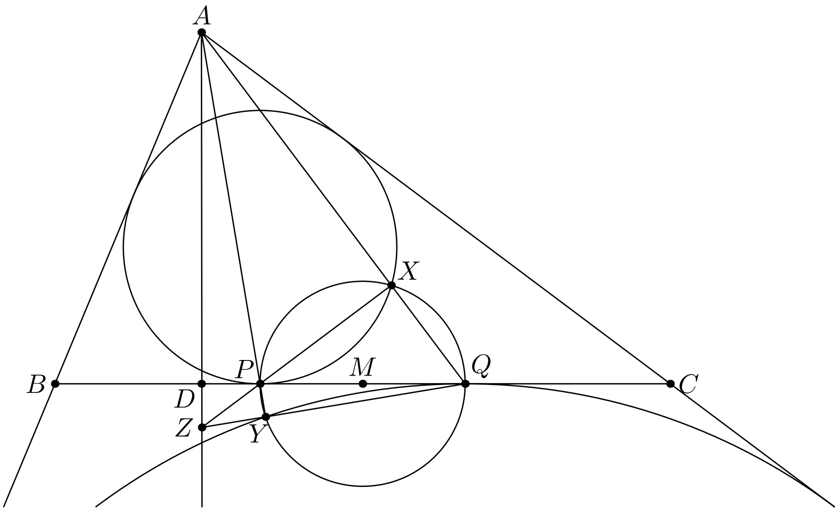
Hence, if $a, b < 29$, our solutions are $(4, 13), (5, 6), (6, 11), (6, 13), (11, 13)$, leading to a (distinct) sum of $17 + 11 + 19 + 24 = 71$. Now, assume that $a = 29$. We just need that $b^2 - 1 \mid 2^3 \cdot 3 \cdot 5 \cdot 7$. We can narrow down the set of all such b to $\{2, 3, 4, 5, 6, 11, 13, 29\}$. All such $a+b$ are greater than 29, and thus have not been counted before. The sum of these b is 73. Since there are 7 solutions, the a components sum to $29 \cdot 8 = 232$, so the sum here is 305.

It follows that the requested sum is $71 + 305 = 376$.

Problem X

In triangle ABC, $AB = 26$, $BC = 42$, and $CA = 40$. Let ω be the incircle of $\triangle ABC$, and let ω_A be the circle tangent to segment BC and the extensions of lines AB and AC past B and C, respectively. Suppose that ω and ω_A are tangent to $\overline{BC}$ at P and Q, respectively, and that X and Y lie on ω and ω_A, respectively, such that $\angle AXP = \angle AYQ = 90°$. If M is the midpoint of $\overline{BC}$ and Z is the intersection of $\overline{PX}$ and $\overline{QY}$, find MZ^2.

Answer. 130



It is well known that $MP = MQ$, so M is the center of Γ, the circle with diameter $\overline{PQ}$. It is well known that if P' and Q' denote the antipodes of P and Q, respectively, on ω and ω_A, respectively, then $P' \in \overline{AQ}$ and $Q' \in \overline{AP}$. Let Ω meet ω and ω_A again at $X' \neq P$ and $Y' \neq Q$, respectively. Since

$$\measuredangle PX'Q = 90° = \measuredangle PX'P' = \measuredangle PX'A \text{ and } \measuredangle PY'Q = 90° = \measuredangle Q'Y'Q = \measuredangle AY'Q,$$

we know that $X = X'$ and $Y = Y'$. Moreover, $X \in \overline{AQ}$ and $Y \in \overline{AZ}$.

Since $\measuredangle AXZ = \measuredangle AYZ = 90°$, Q is the orthocenter of $\triangle APZ$, so $\overline{AZ} \perp \overline{BC}$. Then, let D be the foot from A to $\overline{BC}$, so that $D \in \overline{AZ}$.

Now, we compute some lengths. Scale down by a factor of 2, so that $AB = 13$, $BC = 21$, and $CA = 20$. It is not hard to check that $AD = 12$, $BD = 5$, and $CD = 16$. It follows that $[ABC] = 126$ and the semiperimeter is $s = 27$. Furthermore, $BP = s - AC = 7$, so $DP = 2$. Since $BM = \frac{21}{2}$, $MP = MQ = \frac{7}{2}$, $PQ = 7$, $DQ = 9$, and $AQ = 15$. By Power of a Point from Q on $(ADPX)$,

$$QD \cdot QP = QA \cdot QX \implies 9 \cdot 7 = 15 \cdot QX \implies QX = \frac{21}{5} \implies AX = \frac{54}{5}.$$

By Power of a Point from A on $(DXQZ)$,

$$AX \cdot AQ = AD \cdot AZ \implies \frac{54}{5} \cdot 15 = 12 \cdot AZ \implies AZ = \frac{27}{2} \implies DZ = \frac{3}{2}.$$

Finally, by the Pythagorean Theorem on $\triangle DZM$,

$$ZM = \sqrt{DM^2 + DZ^2} = \sqrt{\left(\frac{11}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \frac{\sqrt{130}}{2}.$$

Restoring the factor of 2 yields $ZM^2 = 130$, the answer.

Problem Y

Suppose that S denotes the set of all positive integers that are divisible by no primes other than 2, 3, and 7, and that $\varphi(n)$ denotes the number of positive integers not exceeding n that are relatively prime to n. Then, there exist relatively prime integers p and q such that

$$\sum_{n\in S}\left(\frac{1}{n^3}\sum_{d|n} d\varphi(d)\right)=\frac{p}{q}.$$

Find $p+q$.

Answer. 919

For convenience, ignore the definitions of p and q in the problem, so that we may use p as a prime freely. Let $P=\{2,3,7\}$, and

$$f(n)=\sum_{d|n} d\varphi(d).$$

Let $\star$ denote Dirichlet convlution. Define $g(d)=1/d$ and $h=\varphi\star g$. Since φ and g are multiplicative, so is h. Notice that $f(n)=n\cdot h(n)$, so f is also multiplicative.

Since f is multiplicative,

$$\begin{aligned}
\sum_{n\in S}\frac{f(n)}{n^3} &= \prod_{p\in P}\left(\sum_{s=0}^{\infty}\frac{f(p^s)}{p^{3s}}\right)\\
&= \prod_{p\in P}\left(\sum_{s=0}^{\infty}\frac{1}{p^{3s}}\cdot\left(\left(\sum_{t=1}^{s}\left[p^{2t}-p^{2t-1}\right]\right)+1\right)\right)\\
&= \prod_{p\in P}\left(\sum_{s=0}^{\infty}\frac{1}{p^{3s}}\cdot\left(\left(p^2\cdot\frac{p^{2s}-1}{p^2-1}-p\cdot\frac{p^{2s}-1}{p^2-1}\right)+1\right)\right)\\
&= \prod_{p\in P}\left(\sum_{s=0}^{\infty}\frac{1}{p^{3s}}\cdot\left(\frac{p^{2s+1}+1}{p+1}\right)\right)\\
&= \prod_{p\in P}\left(\frac{1}{p+1}\left(\sum_{s=0}^{\infty}\left[\frac{p^{2s+1}}{p^{3s}}+\frac{1}{p^{3s}}\right]\right)\right)\\
&= \prod_{p\in P}\left(\frac{1}{p+1}\left(\frac{p}{1-\frac{1}{p}}+\frac{1}{1-\frac{1}{p^3}}\right)\right)\\
&= \prod_{p\in P}\frac{p^3+p^2}{p^3-1}.
\end{aligned}$$

Hence, the value of the summation is

$$\frac{2^3+2^2}{2^3-1}\cdot\frac{3^3+3^2}{3^3-1}\cdot\frac{7^3+7^2}{7^3-1}=\frac{12}{7}\cdot\frac{36}{26}\cdot\frac{392}{342}=\frac{672}{247},$$

and the requested sum is $672+247=919$.

Alternative. An alternate proof of the multiplicity of f without Dirichlet convolutions is as follows: For all n,p,k such that $p\nmid n$,

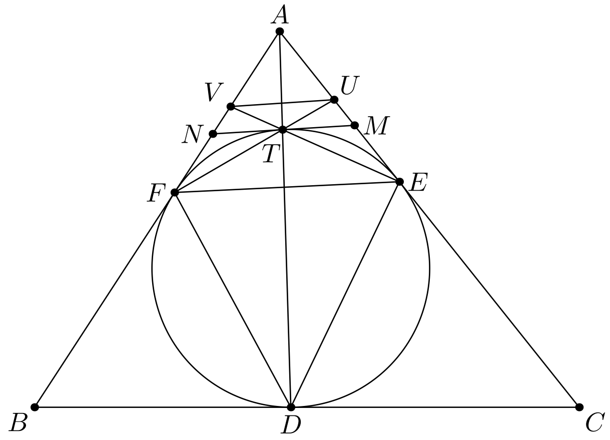
$$\begin{aligned}
f(np^k)=\sum_{a|np^k} a\varphi(a) &= \sum_{a|n}\sum_{s=0}^{k}(ap^s)\varphi(ap^s)=\sum_{a|n} a\varphi(a)\left(\sum_{s=0}^{k}p^s\varphi(p^s)\right)\\
&= \sum_{a|n} a\varphi(a)f(p^k)=f(n)f(p^k).
\end{aligned}$$

Problem Z

In triangle ABC, $AB = 14$, $BC = 17$, and $CA = 15$. The incircle of $\triangle ABC$ touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively; and $\overline{AD}$ meets the incircle of $\triangle ABC$ again at T. Suppose that points M and U lie on $\overline{AC}$ and points N and V lie on $\overline{AB}$ such that $\overline{MN}$ is tangent to the incircle of $\triangle ABC$ at T, and $\overline{EV}$ and $\overline{FU}$ intersect at T. Then, there exist relatively prime positive integers p and q such that $\frac{MU}{NV} = \frac{p}{q}$. Find $p + q$.

Answer. 449

Let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a + b + c)$.



Denote $P = \overline{BC} \cap \overline{EF}$. Since $DETF$ is a harmonic quadrilateral, $P \in \overline{MN}$. By Brianchon's Theorem on $BDCMTN$ and $BCEMNF$, there exists a common point S on $\overline{BM}$, $\overline{CN}$, $\overline{DT}$, and $\overline{EF}$. Now, note that

$$-1 = (A, S; T, D) \overset{P}{=} (A, F; N, B) \tag{1}$$

and

$$-1 = (D, T; E, F) \overset{T}{=} (A, N; V, F). \tag{2}$$

By (1), we can deduce that

$$\frac{NA}{NF} = \frac{BA}{BF} = \frac{c}{s-b}.$$

It follows that

$$\frac{VA}{VN} = \frac{FA}{FN} = 1 + \frac{NA}{NF} = \frac{s-b+c}{s-b}.$$

Moreover we have that

$$\frac{NA}{AF} = \frac{c}{s-b+c} \implies NA = \frac{c(s-a)}{s-b+c}.$$

Thus,

$$\frac{NV}{NA} = \frac{s-b}{2(s-b)+c},$$

so

$$NV = \frac{c(s-a)(s-b)}{\big(2(s-b)+c\big)(s-b+c)} = \frac{14 \cdot 6 \cdot 8}{(2 \cdot 8 + 14)(8 + 14)} = \frac{56}{55}.$$

Similarly,

$$MU = \frac{b(s-a)(s-c)}{\big(2(s-c)+b\big)(s-c+b)} = \frac{15 \cdot 6 \cdot 9}{(2 \cdot 9 + 15)(9 + 15)} = \frac{45}{44},$$

whence $\frac{MU}{NV} = \frac{45}{44} \Big/ \frac{56}{55} = \frac{225}{224}$, and the requested sum is $225 + 224 = 449$.