2019 Mock AIME Tiebreaker Round C

by TheUltimate123 and nukelauncher

You are participating in Tiebreaker Round C because you have tied with another contestant and have a score that is at least 11. The results of this tiebreaker round will determine your placement on the leaderboard.

1 Instructions

1. DO NOT FLIP OVER THIS PAPER UNTIL YOU HAVE STARTED YOUR TIMER.
2. This tiebreaker round is a 5-question, 30-minute examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor penalties for wrong answers. The problems are not necessarily sorted in order of increasing difficulty.
3. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on this exam will require the use of a calculator.
4. Whenever recording an answer for any problem, record the time, including minutes and seconds, remaining on the clock. PM your most recent answers and times for each problem to both TheUltimate123 and nukelauncher.
5. If multiple scores (number of correct answers) tie, then we will look at the times of the latest correct submission. For instance, consider the following hypothetical scenario:

Problem	Answer	Submission	Time Left
1	031	031	4:13
2	068	065	17:18
3	551	981	1:01
4	137	137	26:24
5	111	151	8:19

Problem	Answer	Submission	Time Left
1	031	031	25:11
2	068	068	19:02
3	551	414	13:13
4	137	123	2:19
5	111	110	0:03

In this case, the two contestants have both solved exactly two problems. However, the latest correct submission for the left contestant occurred at 4:13, while that of the right contestant occured at 19:02. Therefore, the right contestant wins.

2 Problems

C1. Call a positive integer *palatable* if when expressed in binary, each contiguous block of zeros that is not a subsequence of another contiguous block of zeros has even length, and each contiguous block of ones that is not a subsequence of another contiguous block of ones has odd length. For example, $57 = 111001_2$ is palatable while $69 = 1000101_2$ is not. Find the number of palatable positive integers N such that $2^{18} < N < 2^{19}$.

C2. Consider all positive integers a, b such that $\operatorname{lcm}(a^2 - 1, b^2 - 1) = 29^2 - 1$. Find the sum of all distinct values of $a + b$.

C3. In triangle ABC, $AB = 26$, $BC = 42$, and $CA = 40$. Let ω be the incircle of $\triangle ABC$, and let ω_A be the circle tangent to segment BC and the extensions of lines AB and AC past B and C, respectively. Suppose that ω and ω_A are tangent to $\overline{BC}$ at P and Q, respectively, and that X and Y lie on ω and ω_A, respectively, such that $\angle AXP = \angle AYQ = 90^\circ$. If M is the midpoint of $\overline{BC}$ and Z is the intersection of $\overline{PX}$ and $\overline{QY}$, find MZ^2.

C4. Suppose that S denotes the set of all positive integers that are divisible by no primes other than 2, 3, and 7, and that $\varphi(n)$ denotes the number of positive integers not exceeding n that are relatively prime to n. Then, there exist relatively prime integers p and q such that

$$\sum_{n \in S} \left(\frac{1}{n^3} \sum_{d \mid n} d\varphi(d) \right) = \frac{p}{q}.$$

Find $p + q$.

C5. In triangle ABC, $AB = 14$, $BC = 17$, and $CA = 15$. The incircle of $\triangle ABC$ touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F, respectively; and $\overline{AD}$ meets the incircle of $\triangle ABC$ again at T. Suppose that points M and U lie on $\overline{AC}$ and points N and V lie on $\overline{AB}$ such that $\overline{MN}$ is tangent to the incircle of $\triangle ABC$ at T, and $\overline{EV}$ and $\overline{FU}$ intersect at T. Then, there exist relatively prime positive integers p and q such that $\frac{MU}{NV} = \frac{p}{q}$. Find $p + q$.