2019 Mock AIME Solutions

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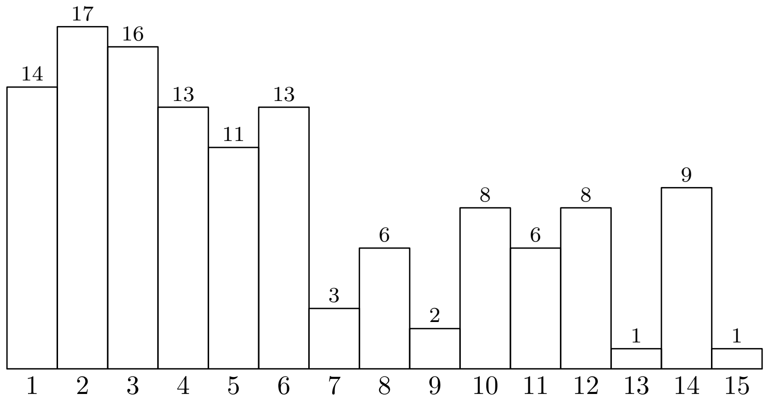
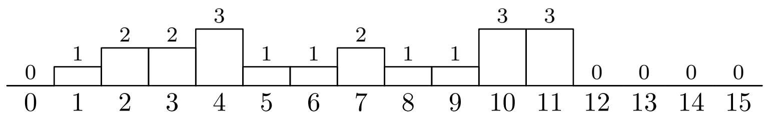
PROBLEM BREAKDOWN

Problem	Author	Answer
1	nukelauncher	250
2	TheUltimate123	035
3	nukelauncher	028
4	nukelauncher	188
5	TheUltimate123	023
6	TheUltimate123	078
7	nukelauncher	113
8	nukelauncher	010
9	TheUltimate123	322
10	TheUltimate123	519
11	TheUltimate123	068
12	nukelauncher	264
13	TheUltimate123	580
14	nukelauncher	229
15	TheUltimate123	521

PROBLEM SOLVE-RATE



SCORE DISTRIBUTION



Problem 1

Suppose that 64 teams, labeled T_1, T_2, T_3, $\ldots$, T_{64}, are participating in a tournament. For all $1 \le i \le 64$, team i initially has i players. The teams play a series of matches to determine a winner. A match involves two players from different teams, and will result in one player winning and the other losing; no ties occur. When a player loses, he is eliminated, and when all players of a team are eliminated, the team is eliminated. After exactly 2019 games, T_k is crowned the champion. Find the sum of all possible values of k.

Answer. 250

First, there are

$$1 + 2 + 3 + \cdots + 64 = \frac{64 \cdot 65}{2} = 2080$$

players in the tournament. For T_k to win, at least $2080 - k$ players must have been eliminated. Since each match eliminates exactly one player,

$$2019 \ge 2080 - k \implies k \ge 61.$$

Hence, the requested sum is $61 + 62 + 63 + 64 = 250$.

Problem 2

There are positive real numbers x, y, z such that $\log_x(yz) = 59$ and $\log_y(zx) = 89$. Find $\log_{xy}(z)$.

Answer. 035

First solution. We are given that $x^{59} = yz$ and $y^{89} = zx$. Then,

$$z = \frac{x^{59}}{y} = \frac{y^{89}}{x},$$

and multiplying gives $x^{60} = y^{90}$, or $x^2 = y^3$. Thus, there exists a real number t such that $x = t^3$ and $y = t^2$. Furthermore,

$$z = \frac{(t^3)^{59}}{t^2} = t^{175},$$

whence

$$\log_{xy}(z) = \frac{\log_t z}{\log_t x + \log_t y} = \frac{175}{3+2} = 35,$$

the answer.

Second solution. Let $a = 59$ and $b = 89$. We have that $x^a = yz$ and $y^b = zx$. The latter gives $x = y^b z^{-1}$, so plugging into the first, $y^{ab} z^{-a} = yz$, whence $y^{ab-1} = z^{a+1}$. Then,

$$\log_z y = \frac{a+1}{ab-1} \text{ and } \log_z x = \frac{ab+b}{ab-1} - 1 = \frac{b+1}{ab-1}.$$

It follows that

$$\frac{1}{\log_{xy} z} = \log_z x + \log_z y = \frac{b+1}{ab-1} + \frac{a+1}{ab-1} = \frac{a+b+2}{ab-1} = \frac{1}{35},$$

and the answer is 35.

Problem 3

Find the sum of all positive integers n such that $n^2 + 13$ is divisible by $2n - 1$.

Answer. 028

First solution. Since $2n - 1$ is odd, the problem is equivalent to

$$2n - 1 \mid 4(n^2 + 13) = 4n^2 + 52 = (2n - 1)(2n + 1) + 53,$$

whence $2n - 1 \mid 53$. It follows that $2n - 1 = 1$ or $2n - 1 = 53$, which yield $n = 1$ and $n = 27$, respectively. The requested sum is $1 + 27 = 28$.

Second solution. Note that $n = 1$ works. However, for all $n > 1$, it is possible for $2n - 1$ to divide $n^2 + 13$ only if they are not relatively prime. By the Euclidean Algorithm,

$$\gcd(n^2 + 13, 2n - 1) = \gcd(2n^2 + 26, 2n - 1) = \gcd(n + 26, 2n - 1) = \gcd(n + 26, 53).$$

It follows that $53 \mid n + 26$, so $n = 53k - 26$. However, $2n - 1 \mid n^2 + 13$ only if

$$2n - 1 = \gcd(n^2 + 13, 2n - 1) = \gcd(n + 26, 53) \le 53,$$

or $n \le 27$. Thus, the only valid n of the form $53k - 26$ occurs when $k = 1$, or $n = 27$. Consequently the requested sum is $1 + 27 = 28$.

Problem 4

Given that

$$\sum_{k=1}^{\infty} \frac{\sin(\frac{k\pi}{6})}{3^k} = \frac{a+b\sqrt{c}}{d},$$

where a, b, c, d are positive integers such that a, b, and d share no prime factor and c is not divisible by the square of any prime, find $a+b+c+d$.

Answer. 188

First solution. Let

$$C = \sum_{k=0}^{\infty} \frac{\cos(\frac{k\pi}{6})}{3^k} \text{ and } S = \sum_{k=0}^{\infty} \frac{\sin(\frac{k\pi}{6})}{3^k}.$$

Since the expression for $k=0$ is 0, we seek S. However,

$$C + iS = \sum_{k=0}^{\infty} \frac{\cos(\frac{k\pi}{6}) + i\sin(\frac{k\pi}{6})}{3^k} = \sum_{k=0}^{\infty} \frac{e^{ki\pi/6}}{3^k} = \frac{1}{1-\frac{1}{3}e^{i\pi/6}}.$$

Substituting $e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$,

$$C + iS = \left(\frac{6-\sqrt{3}}{6} - \frac{1}{6}i\right)^{-1} = \frac{6}{(6-\sqrt{3})-i} = \frac{18-3\sqrt{3}+3i}{20-6\sqrt{3}},$$

whence

$$S = \operatorname{Im}(C+iS) = \frac{3}{20-6\sqrt{3}} = \frac{30+9\sqrt{3}}{146},$$

and the requested sum is $30+9+3+146 = 188$.

Second solution. Since $\cos(\pi+\theta) = -\cos\theta$ and $\cos(\pi+2\theta) = \cos\theta$,

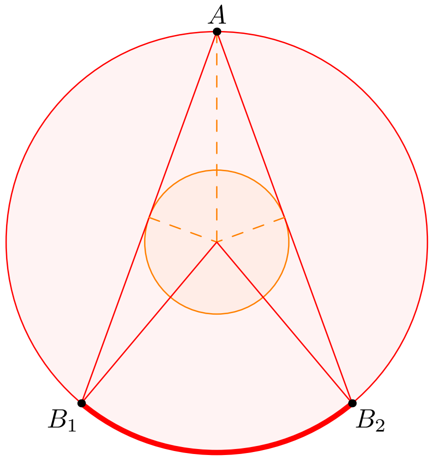
$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\sin(\frac{k\pi}{6})}{3^k} &= \sum_{k=1}^{5} \sin\left(\frac{k\pi}{6}\right) \sum_{i=0}^{\infty} \left(\frac{(-1)^i}{3^{6i+k}}\right) \\
&= \sum_{k=1}^{5} \sin\left(\frac{k\pi}{6}\right) \left(\frac{3^{-k}}{1+3^{-6}}\right) \\
&= \frac{729}{730} \sum_{k=1}^{5} 3^{-k} \sin\left(\frac{k\pi}{6}\right) \\
&= \frac{729}{730} \left(\frac{1}{2\cdot 3} + \frac{\sqrt{3}}{2\cdot 3^2} + \frac{1}{3^3} + \frac{\sqrt{3}}{2\cdot 3^4} + \frac{1}{2\cdot 3^5}\right) \\
&= \frac{243+81\sqrt{3}+54+9\sqrt{3}+3}{1460} \\
&= \frac{30+9\sqrt{3}}{146},
\end{aligned}$$

and the requested sum is $30+9+3+146 = 188$.

Problem 5

Points A and B are randomly and uniformly chosen on the circumference of the circle $x^2+y^2=1$. Find the expected number of ordered pairs of real numbers (p,q) such that the point (p,q) lies on line AB and there exists an integer $1 \le k \le 45$ such that $p^2+q^2=\sin^2(k^\circ)$.

Answer. 023



Let O denote the origin. Suppose that ω_k denotes the circle centered at O with radius $\sin k$. The problem is simply asking for the total number of times AB intersects ω_k for all integers $1 \le k \le 45$. We look at each k independently. On the unit circle, fix point A first.

Consider the locus of all B such that $\overline{AB}$ intersects ω_k. It is easy to check that this is the arc $\widehat{B_1B_2}$ not containing A such that $\overline{AB_1}$ and $\overline{AB_2}$ are tangent to ω_k. This is because for every point P on ω_k, the ray AP lies within $\angle B_1AB_2$, and every ray within $\angle B_1AB_2$ must intersect ω_k.

If X denotes the point where $\overline{AB_1}$ touches ω_k, it is easy to see that

$$\widehat{B_1B_2} = 2\angle B_1AB_2 = 4\angle XAO = 4\arcsin\left(\frac{OX}{OA}\right) = 4k.$$

Hence, the probability a given line AB intersects ω_k is $\frac{4k}{360}=\frac{k}{90}$. The probability that $\overline{AB}$ is tangent to ω_k is infinitesimal and negligible, and if $\overline{AB}$ intersects ω_k but is not tangent to ω_k, then AB intersects ω_k at two points. Hence, the expected number of times $\overline{AB}$ intersects ω_k is $\frac{2k}{90}=\frac{k}{45}$. Summing over all such k, the answer is

$$\sum_{i=1}^{45}\frac{k}{45} = \frac{45\cdot 46}{2\cdot 45} = 23,$$

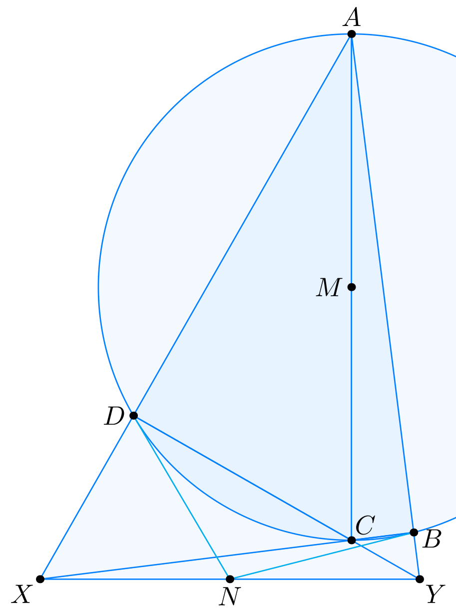
and we are done.

Problem 6

In cyclic quadrilateral $ABCD$, $AB=8$, $BC=1$, $CD=4$, and $DA=7$. If M denotes the midpoint of $\overline{AC}$, X the intersection of lines AD and BC, Y the intersection of lines AB and CD, and N the midpoint of $\overline{XY}$, then MN can be expressed in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime. Find $p+q+r$.

Answer. 078

First solution. It is easy to determine that $\overline{AC}$ is a diameter of $(ABCD)$, and that $AC=\sqrt{65}$. The key observation is that A,X,Y,C form an orthocentric system, a corollary of which is that $\overline{AC}\perp\overline{XY}$.



First, let $XC=s, XD=t, YC=u, YB=v$. Then, since $\triangle XAB\sim\triangle XCD$, we have that

$$2=\frac{s+1}{t}=\frac{t+7}{s}.$$

Solving, $s=5$ and $t=3$. Similarly, $\triangle YAD\sim\triangle YCB$, so

$$7=\frac{u+4}{v}=\frac{v+8}{u}.$$

Solving, $u=\frac{5}{4}$ and $v=\frac{3}{4}$. It follows that by the Pythagorean theorem on either $\triangle XDY$ or $\triangle XBY$, we compute that $XY=\frac{3\sqrt{65}}{4}$.

It is well-known that $\overline{NB}$ and $\overline{ND}$ are tangent to $(ABCD)$. Then,

$$\begin{aligned} MN^2-\left(\frac{1}{2}AC\right)^2 &= \operatorname{Pow}_{(ABCD)}(N)=NB^2=\left(\frac{1}{2}XY\right)^2 \\ \implies MN^2 &= \left(\frac{\sqrt{65}}{2}\right)^2+\left(\frac{3\sqrt{65}}{8}\right)^2=\frac{25\cdot 65}{64} \\ \implies MN &= \frac{5\sqrt{65}}{8}, \end{aligned}$$

and the requested sum is $5+65+8=78$.

Second solution. Like above, note that $\overline{AC}$ is a diameter of $(ABCD)$, and that $AC=\sqrt{65}$. Furthermore, C is the orthocenter of $\triangle AXY$. Let $Z=\overline{AC}\cap\overline{XY}$ be the foot from A to $\overline{XY}$, and let $\alpha=\angle CAD$ and $\beta=\angle CAB$. It is easy to check that $\tan\alpha=\frac{4}{7}$ and $\tan\beta=\frac{1}{8}$. Then,

$$\tan(\alpha+\beta)=\frac{\frac{4}{7}+\frac{1}{8}}{1-\frac{4}{7}\cdot\frac{1}{8}}=\frac{3}{4}.$$

Hence,

$$AX = AB\sec(\alpha+\beta) = 8\cdot\frac{5}{4} = 10$$

and similarly $AY = \frac{35}{4}$. Now,

$$AZ = AX\cos\alpha = 10\cdot\frac{7}{\sqrt{65}} = \frac{14\sqrt{65}}{13}$$

and

$$XZ = AX\sin\alpha = 10\cdot\frac{4}{\sqrt{65}} = \frac{8\sqrt{65}}{13}.$$

Similarly, $YZ = \frac{7\sqrt{65}}{52}$. It follows that

$$MZ = AZ - AM = \frac{14\sqrt{65}}{13} - \frac{\sqrt{65}}{2} = \frac{15\sqrt{65}}{26}.$$

Furthermore,

$$NZ = XZ - \frac{XY}{2} = \frac{8\sqrt{65}}{13} - \frac{3\sqrt{65}}{8} = \frac{25\sqrt{65}}{104}.$$

It follows that

$$MN = \sqrt{MZ^2+NZ^2} = \frac{5\sqrt{65}}{104}\sqrt{12^2+5^2} = \frac{5\sqrt{65}}{8},$$

and the requested sum is $5+65+8=78$.

Problem 7

There are N non-congruent rectangles $\mathcal{R}$ in the coordinate plane such that all of $\mathcal{R}$'s vertices have integer coordinates and $\mathcal{R}$'s area is 32400. Find N.

Answer. 113

Let the sides of $\mathcal{R}$ be a and b, so that $ab = 32400 = 2^4 \cdot 3^4 \cdot 5^2$. Then, $a^2b^2 = 2^8 \cdot 3^8 \cdot 5^4$. Since $2 = 1^2 + 1^2$, $5 = 1^2 + 2^2$, and $10 = 1^2 + 3^2$, if a and b have odd exponents of 2 or 5, we can orient the rectangle such that the edges have factors of $\sqrt{2}$, $\sqrt{5}$, or $\sqrt{10}$. Hence, there are no restrictions on the exponents of 2 and 5 in a and b.

However, since quadratic residues are $0, 1$ modulo 3, if $\gcd(m, n) = 1$, then $m^2 + n^2 \equiv 1, 2 \pmod{3}$, so it is impossible to orient a rectangle such that the sides have factors of $\sqrt{3k}$, where $3 \nmid k$. Hence, the exponents of 3 must be even.

Of the $9 \cdot 5 \cdot 5$ ordered pairs (a, b), one has $a = b$ and the rest have $a \neq b$; hence, to avoid overcount, we halve the cases where $a \neq b$. Hence, the number of such $\mathcal{R}$ is

$$N = \frac{9 \cdot 5 \cdot 5 - 1}{2} + 1 = 113,$$

the answer.

Problem 8

Let a be the smallest real number such that the roots of the polynomial

$$P(x) = ax^3 - x^2 - (a^2+1)x + a^2 - 1$$

are all real. Then, the largest of these roots can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer. 010

We know that

$$0 = -P(x) = (x-1)a^2 - x^3a + (x^2+x+1).$$

Applying the quadratic formula with respect to a, we find that

$$a = \frac{x^3 \pm \sqrt{x^6 - 4(x-1)(x^2+x+1)}}{2(x-1)} = \frac{x^3 \pm \sqrt{x^6 - 4x^3 + 4}}{2(x-1)} = \frac{x^3 \pm (x^3-2)}{2(x-1)}.$$

Hence, $a = \frac{1}{x-1}$ or $a = x^2+x+1$. However, $a = x^2+x+1$ only has roots for x if $a \geq \frac{3}{4}$, so the smallest such a is $\frac{3}{4}$. Solving for x, we find that $x \in \{\frac{7}{3}, -\frac{1}{2}\}$, so the largest root of P is $\frac{7}{3}$, and the requested sum is $7+3=10$.

Remark. The values of a in terms of x do in fact imply that our polynomial can be factored as

$$P(x) = \big(ax - (a+1)\big)\big(x^2 + x - (a-1)\big).$$

Problem 9

For relatively prime positive integers a and b and positive real numbers c and θ, let K denote the area of the triangle with sides of length $a\sin\theta$, $b\cos\theta$, and $c\tan\theta$, given that it is positive. Suppose that if a and b remain fixed, and c and θ vary, then K achieves a maximum when $c = 85$. Find the sum of all distinct possible values of $a+b$.

Answer. 322

Let ζ be the angle opposite the side of length $c\tan\theta$. Then,

$$K = \frac{a\sin\theta \cdot b\cos\theta \cdot \sin\zeta}{2} = \frac{ab\sin 2\theta \sin\zeta}{4} \le \frac{ab}{4},$$

with equality when $\theta = 45^\circ$ and $\zeta = 90^\circ$. Then, by the Pythagorean Theorem,

$$\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{b}{\sqrt{2}}\right)^2 = c^2 \implies a^2 + b^2 = 2c^2.$$

Note that if $m = \frac{1}{2}(a+b)$ and $n = \frac{1}{2}(a-b)$, then

$$2c^2 = a^2 + b^2 = (m+n)^2 + (m-n)^2 = 2(m^2+n^2),$$

so $m^2 + n^2 = c^2 = 85^2$. Since $a+b$ and $a-b$ have the same parity, m and n are either both integers or both half of odd integers.

We claim that the latter case is impossible. Assume for the sake of contradiction that m and n are both half of odd integers. Let $m' = 2m$ and $n' = 2n$, so that

$$4 \cdot 85^2 = (m')^2 + (n')^2 \equiv 2 \pmod 4,$$

a contradiction. Hence, $m, n \in \mathbb{Z}$. Because $\gcd(m,n) \mid \gcd(a,b) = 1$, m and n are relatively prime, so $(m, n, 85)$ is a primitive Pythagorean triple in which $m > n$.

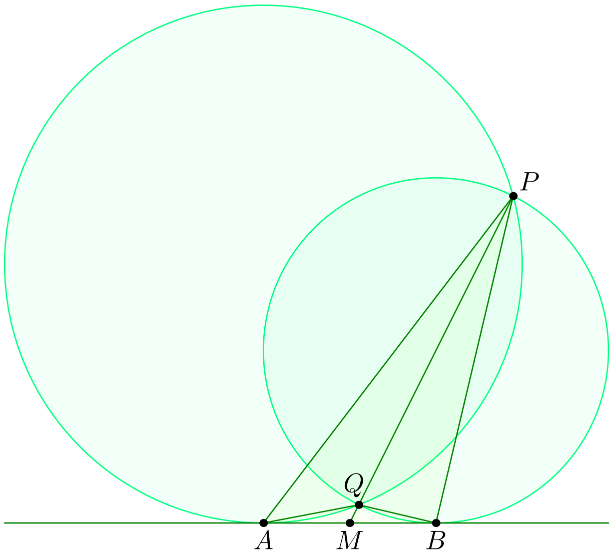
It is well known that for positive integers $p > q$, $(p^2 - q^2, 2pq, p^2 + q^2)$ generates all primitive Pythagorean triples. It is not hard to check that the only solutions to $p^2 + q^2 = 85$ are $(7,6)$ and $(9,2)$. These yield the Pythagorean triples $(84, 13, 85)$ and $(77, 36, 85)$. Since $2m = a+b$, the requested sum is $2(84+77) = 322$.

Remark. The triples $(84, 13, 85)$ and $(77, 36, 85)$ yield the solutions $(a,b) = (97, 71)$ and $(113, 41)$, respectively.

Problem 10

Suppose that circles Ω_1 and Ω_2 intersect at P and Q, and that line AB is tangent to Ω_1 and Ω_2 at A and B, respectively, such that Q is closer to $\overline{AB}$ than P. If $AB = 2$, $PA = 20$, and $PB = 19$, then $QA \cdot QB$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

Answer. 519



First, remark that if in any triangle XYZ, $YZ = 2$ and N denotes the midpoint of $\overline{YZ}$, then by Stewart's Theorem,

$$XN^2 = \frac{XY^2 + XZ^2}{2} - 1.$$

Let $M = \overline{AB} \cap \overline{PQ}$. Then, $MA^2 = MP \cdot MQ = MB^2$, so M is the midpoint of $\overline{AB}$. By the Tangency Criterion, $\triangle MAQ \sim \triangle MPA$ and $\triangle MBQ \sim \triangle MPB$. Hence,

$$\frac{PA}{QA} = \frac{MP}{MA} = \frac{MP}{MB} = \frac{PB}{QB}.$$

It follows that there exists a real number t such that $QA = 20t$ and $QB = 19t$. By Power of a Point from M, $MP \cdot MQ = 1$. If $u = \frac{20^2+19^2}{2} = \frac{761}{2}$, using our remark,

$$1 = MP^2 \cdot MQ^2 = \left(\frac{20^2 + 19^2}{2} - 1\right)\left(t^2 \cdot \frac{20^2 + 19^2}{2} - 1\right) = (u-1)(t^2u - 1),$$

whence

$$1 + \frac{1}{t^2} = u = \frac{761}{2} \implies t^2 = \frac{2}{759} \implies QA \cdot QB = 380t^2 = \frac{760}{759},$$

and the requested remainder is $760 + 759 \equiv 519 \pmod{1000}$.

Problem 11

Adam, Bob, and Charlie each flip a coin every day, starting from Day 1, until all three of them have flipped heads at least once. The last of them to flip heads for the first time does so on Day X. The probability that X is even can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Answer. 068

If k remains fixed, note that

$$\begin{aligned}\mathbb{P}(X>k) &= 1-\mathbb{P}(\text{All three have flipped heads in the first } k \text{ days})\\ &= 1-\mathbb{P}(\text{Adam has flipped heads in the first } k \text{ days})^3\\ &= 1-\big(1-\mathbb{P}(\text{Adam has not flipped heads in the first } k \text{ days})\big)^3\\ &= 1-\left(1-\frac{1}{2^k}\right)^3.\end{aligned}$$

It follows that

$$\mathbb{P}(X=k)=\mathbb{P}(X>k-1)-\mathbb{P}(X>k)=\left(1-\frac{1}{2^k}\right)^3-\left(1-\frac{1}{2^{k-1}}\right)^3.$$

Hence,

$$\begin{aligned}\mathbb{P}(2\mid X) &= \sum_{i=1}^{\infty}\mathbb{P}(X=2i)=\sum_{i=1}^{\infty}\left(\left(1-\frac{1}{2^{2i}}\right)^3-\left(1-\frac{1}{2^{2i-1}}\right)^3\right)\\ &= \sum_{i=1}^{\infty}\left(-3\left(-\frac{1}{2^{2i-1}}+\frac{1}{2^{2i}}\right)+3\left(-\left(\frac{1}{2^{2i-1}}\right)^2+\left(\frac{1}{2^{2i}}\right)^2\right)\right.\\ &\qquad\qquad\left.-\left(-\left(\frac{1}{2^{2i-1}}\right)^3+\left(\frac{1}{2^{2i}}\right)^3\right)\right)\\ &= -3\left(\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^k\right)+3\left(\sum_{k=1}^{\infty}\left(-\frac{1}{2^2}\right)^k\right)-\sum_{k=1}^{\infty}\left(-\frac{1}{2^3}\right)^k\\ &= -3\left(\frac{-1/2}{3/2}\right)+3\left(\frac{-1/4}{5/4}\right)-\frac{-1/8}{9/8}=1-\frac{3}{5}+\frac{1}{9}=\frac{23}{45},\end{aligned}$$

and the requested sum is $23+45=68$.

Problem 12

There are nonzero real numbers x, y, z such that

$$\begin{aligned} 0 &= x^2y + 2x - y \\ &= y^2z + 2y - z \\ &= z^2x + 2z - x. \end{aligned}$$

Let T denote the least possible value of $100|xyz|$. Find the greatest integer that does not exceed T.

Answer. 264

Rewrite this as

$$y = \frac{2x}{1-x^2}, \quad z = \frac{2y}{1-y^2}, \quad x = \frac{2z}{1-z^2}.$$

Suppose that $x = \tan\alpha$. Then, by the Tangent Double Angle Formula, $y = \tan 2\alpha$, $z = \tan 4\alpha$, and $x = \tan 8\alpha$. Hence, $\tan\alpha = \tan 8\alpha$, so $\alpha = \frac{k\pi}{7}$ for some integer $0 < k < 7$. It then follows that

$$\{x, y, z\} = \left\{\frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7}\right\} \text{ or } \left\{\frac{3\pi}{7}, \frac{6\pi}{7}, \frac{5\pi}{7}\right\}.$$

Since $\tan\theta = -\tan(\pi - \theta)$, the values of xyz for the two solutions are additive inverses, so

$$x^2y^2z^2 = -\prod_{k=1}^{6} \tan\left(\frac{k\pi}{7}\right).$$

It is easy to see that for all $\theta \in \{\frac{\pi}{7}, \frac{2\pi}{7}, \ldots, \frac{6\pi}{7}\}$, if $t = \tan\theta$,

$$\frac{3t - t^3}{1 - 3t^2} = \tan 3\theta = -\tan 4\theta = -\frac{4t - 4t^3}{1 - 6t^2 + t^4}.$$

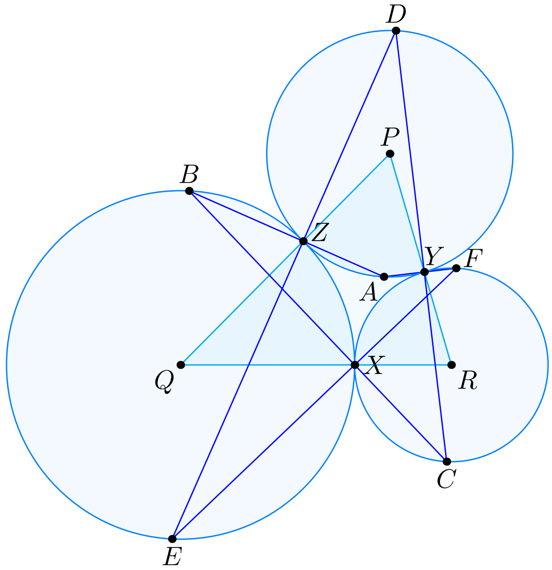
After removing a factor of t and expanding, the polynomial will have degree 6 and leading coefficient 1. The roots of the resulting polynomial must be $\tan(\frac{\pi}{7}), \tan(\frac{2\pi}{7}), \ldots, \tan(\frac{6\pi}{7})$, so we only need the constant term. Indeed, the constant term is 7, so the product of the roots is -7, and so $x^2y^2z^2 = 7$. It follows that $|xyz| = \sqrt{7}$, and $\lfloor 100|xyz| \rfloor = \lfloor 100\sqrt{7} \rfloor = 264$, the answer.

Problem 13

Circles Γ_1, Γ_2, and Γ_3 are pairwise externally tangent and have diameters of length 70, 99, and 55, respectively. Suppose that Γ_2 and Γ_3 touch at X, Γ_3 and Γ_1 touch at Y, and Γ_1 and Γ_2 touch at Z. A point A is chosen on the minor arc YZ of Γ_1. Ray AZ intersects Γ_2 again at B, ray BX intersects Γ_3 again at C, ray CY intersects Γ_1 again at D, ray DZ intersects Γ_2 again at E, ray EX intersects Γ_3 again at F, and ray FY intersects Γ_1 again at G. Find the maximum possible value of $AB + BC + CD + DE + EF + FG$.

Answer. 580

Let the centers of Γ_1, Γ_2, Γ_3 be P, Q, R, respectively. By homothety, $\overline{PA}$ and $\overline{QB}$ are parallel but in opposite directions; similarly so are $\overline{QB}$, $\overline{CR}$ and $\overline{CR}$, $\overline{PD}$. Thus $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are diameters, and $G = A$.



Let $a = 70$, $b = 99$, $c = 55$ be the diameters of Γ_1, Γ_2, Γ_3, respectively, and also let $2\theta = \widehat{BX} = \widehat{CX}$. It follows that if $2\beta = \angle Q$ and $2\gamma = \angle R$, then $\widehat{AZ} = \widehat{BZ} = 2(\theta - \beta)$ and $\widehat{CY} = \widehat{DY} = 2(\theta + \gamma)$. Hence, by the Extended Law of Sines,

$$AB = AZ + BZ = (a + b)\sin(\theta - \beta).$$

Similarly, $BC = (b + c)\sin\theta$ and $CD = (c + a)\sin(\theta + \gamma)$. Remark that

$$DE = (a + b)\cos(\theta - \beta),$$

and similarly, $EF = (b + c)\cos\theta$ and $FA = (c + a)\cos(\theta + \gamma)$.

We now compute β and γ. Scale $\triangle PQR$ up by a factor of 2, so that angles are preserved. It is easy to determine that $PQ = 169$, $QR = 154$, $RP = 125$, and if S denotes the foot from P to $\overline{QR}$, $PS = 120$, $QS = 119$, $RS = 35$. Then, $\cos Q = \frac{119}{169}$ and $\cos R = \frac{7}{25}$. By the Half Angle Formulas,

$$\sin\beta = \tfrac{5}{13},\ \cos\beta = \tfrac{12}{13},\ \sin\gamma = \tfrac{3}{5},\ \cos\gamma = \tfrac{4}{5},$$

and by the Cauchy-Schwarz Inequality (after scaling back up),

$$\begin{aligned}
&AB + BC + CD + DE + EF + FA \\
&= 169\sin(\theta - \beta) + 154\sin\theta + 125\sin(\theta + \gamma) \\
&\qquad + 169\cos(\theta - \beta) + 154\cos\theta + 125\cos(\theta + \gamma) \\
&= (156\sin\theta - 65\cos\theta) + 154\sin\theta + (100\sin\theta + 75\cos\theta) \\
&\qquad + (156\cos\theta + 65\sin\theta) + 154\cos\theta + (100\cos\theta - 75\sin\theta) \\
&= 420\sin\theta + 400\cos\theta = 20(21\sin\theta + 20\cos\theta) \\
&\le 20\sqrt{(21^2 + 20^2)(\sin^2\theta + \cos^2\theta)} = 580,
\end{aligned}$$

which is easily achievable by taking the equality case of the Cauchy-Schwarz Inequality (shown in the diagram).

Remark. In the above solution, we use the Cauchy-Schwarz Inequality to maximize the expression $21\sin\theta + 20\cos\theta$. Alternatively, note that if $\zeta = \tan^{-1}(\frac{20}{21})$,

$$21\sin\theta + 20\cos\theta = 29(\cos\zeta\sin\theta + \sin\zeta\cos\theta) = 29\sin(\zeta+\theta) \le 29,$$

with equality easily achievable.

Remark. It appears that $\overline{AP} \perp \overline{QR}$, but this is not true. In fact, if we solve the equality case for θ, we find that $\theta = 2\tan^{-1}(\frac{3}{7})$. Hence, if S is the foot from P to $\overline{QR}$, then (oriented clockwise)

$$\angle APS = 4\tan^{-1}\left(\frac{3}{7}\right) - 90^\circ \approx 2.79436205459274^\circ.$$

Problem 14

Consider all positive integers k for which there exists a positive integer n such that

$$n^4 + \frac{n^3+n^2}{2} + n + 1 = k^2.$$

Find the greatest of all such k.

Answer. 229

Notice that for all n,

$$\left(n^2 + \left\lfloor \frac{n}{4} \right\rfloor\right)^2 \leq \left(n^2 + \frac{n}{4}\right)^2 = n^4 + \frac{n^3}{2} + \frac{n^2}{16} < n^4 + \frac{n^3+n^2}{2} + n + 1.$$

Furthermore, if $n \geq 15$,

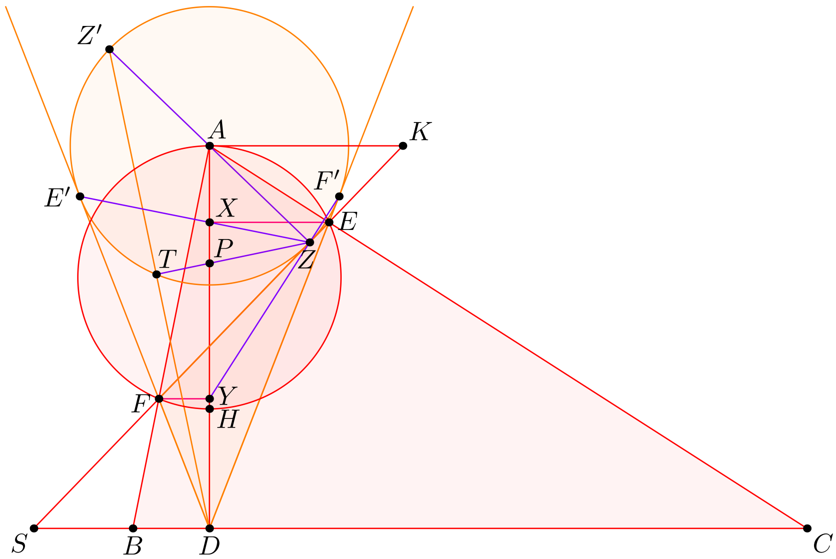
$$\begin{aligned}\left(n^2 + \left\lfloor \frac{n}{4} \right\rfloor + 1\right)^2 \geq \left(n^2 + \frac{n+1}{4}\right)^2 &= n^4 + \frac{n^3}{2} + \frac{9n^2}{16} + \frac{n}{8} + \frac{1}{16} \\ &= n^4 + \frac{n^3+n^2}{2} + n + 1 + \frac{1}{16}(n-15)(n+1) \\ &\geq n^4 + \frac{n^3+n^2}{2} + n + 1,\end{aligned}$$

with equality iff $n = 15$. Hence, k is maximized when $n = 15$, at which $k = n^2 + \lfloor \frac{n}{4} \rfloor + 1 = 229$, and we are done.

Problem 15

In triangle ABC, $AB = 11$, $BC = 19$, and $CA = 20$. Let O denote the circumcenter of $\triangle ABC$, and D, E, and F denote the feet of the altitudes from A, B, and C, respectively. Points X and Y are the feet of the perpendiculars from E and F, respectively, to $\overline{AD}$. If $\overline{AO}$ intersects $\overline{EF}$ at Z, then there exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. Suppose that $\overline{ZT}$ intersects $\overline{AD}$ at P. Then, there exist relatively prime positive integers m and n such that $\frac{PX}{PY} = \frac{m}{n}$. Find $m+n$.

Answer. 521



Since wrt. $\angle A$, $\overline{BC}$ and $\overline{EF}$ are antiparallel, and furthermore O and H are isogonal conjugates wrt. $\triangle ABC$, Z is the projection of A onto $\overline{EF}$. Since $\triangle ABC$ is the excentral triangle of $\triangle DEF$, Z is the D-extouch point of $\triangle DEF$. Denote by E' and F' the points where the D-excircle, ω_D, touches $\overline{DF}$ and $\overline{DE}$, respectively.

Let Z' be the antipode of Z on ω_D. Then, since $\measuredangle DTZ = 90^\circ = \measuredangle Z'TZ$, we have that D, T, Z' are collinear. Since $AXZE$ is cyclic,

$$\measuredangle AZX = \measuredangle AEX = \measuredangle ACB = \measuredangle EFA = 90^\circ - \measuredangle FAZ = \measuredangle AZE',$$

so Z, X, E' are collinear. Similarly, Z, Y, F' are collinear. It follows that

$$-1 = (E', F'; Z', T) \stackrel{Z}{=} (X, Y; A, P).$$

Hence,

$$\frac{PX}{PY} = \frac{AX}{AY} = \frac{AE \sin C}{AF \sin B} = \left(\frac{AB}{AC}\right)^2 = \frac{121}{400},$$

and the requested sum is $121 + 400 = 521$.

Remark. If P_∞ denotes the point at infinity along $\overline{BC}$ and P' denotes the point on $\overline{EF}$ such that $\overline{BC} \parallel \overline{PP'}$, then

$$-1 = (X, Y; A, P) \stackrel{P_\infty}{=} (E, F; K, P').$$

However, it is known that $\overline{KA}$ is tangent to (AEF), so $\overline{AP'}$ is the A-symmedian of $\triangle AEF$. Since $\overline{BC}$ and $\overline{EF}$ are antiparllel wrt. $\angle A$, line AP' bisects $\overline{BC}$.