MOCK AMC

TheUltimate123 & nukelauncher

American Mathematics Competitions

2nd Annual

Mock AIME

Mock American Invitational Mathematics Examination

Sunday, April 21, 2019

1. DO NOT OPEN THIS BOOKLET UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor penalties for wrong answers.
3. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on this exam will require the use of a calculator.
4. A combination of your AIME score and your American Mathematics Contest 12 score is not used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of your AIME score and your American Mathematics Contest 10 score is not used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO and USAJMO will not be given.
5. PM all of your answers to both TheUltimate123 and nukelauncher. Do not edit your original message, or your score will be discarded.

   The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

1. Suppose that 64 teams, labeled $T_1, T_2, T_3, \ldots, T_{64}$, are participating in a tournament. For all $1 \leq i \leq 64$, team i initially has i players. The teams play a series of matches to determine a winner. A match involves two players from different teams, and will result in one player winning and the other losing; no ties occur. When a player loses, he is eliminated, and when all players of a team are eliminated, the team is eliminated. After exactly 2019 games, T_k is crowned the champion. Find the sum of all possible values of k.

2. There are positive real numbers x, y, z such that $\log_x(yz) = 59$ and $\log_y(zx) = 89$. Find $\log_{xy}(z)$.

3. Find the sum of all positive integers n such that $n^2 + 13$ is divisible by $2n - 1$.

4. Given that
$$\sum_{k=1}^{\infty} \frac{\sin(\frac{k\pi}{6})}{3^k} = \frac{a + b\sqrt{c}}{d},$$
where a, b, c, d are positive integers such that a, b, and d share no prime factor and c is not divisible by the square of any prime, find $a + b + c + d$.

5. Points A and B are randomly and uniformly chosen on the circumference of the circle $x^2 + y^2 = 1$. Find the expected number of ordered pairs of real numbers (p, q) such that the point (p, q) lies on line AB and there exists an integer $1 \leq k \leq 45$ such that $p^2 + q^2 = \sin^2(k^\circ)$.

6. In cyclic quadrilateral $ABCD$, $AB = 8$, $BC = 1$, $CD = 4$, and $DA = 7$. If M denotes the midpoint of $\overline{AC}$, X the intersection of lines AD and BC, Y the intersection of lines AB and CD, and N the midpoint of $\overline{XY}$, then MN can be expressed in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime. Find $p + q + r$.

7. There are N non-congruent rectangles $\mathcal{R}$ in the coordinate plane such that all of $\mathcal{R}$'s vertices have integer coordinates and $\mathcal{R}$'s area is 32400. Find N.

8. Let a be the smallest real number such that the roots of the polynomial
$$P(x) = ax^3 - x^2 - (a^2 + 1)x + a^2 - 1$$
are all real. Then, the largest of these roots can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. For relatively prime positive integers a and b and positive real numbers c and θ, let K denote the area of the triangle with sides of length $a \sin\theta$, $b \cos\theta$, and $c \tan\theta$, given that it is positive. Suppose that if a and b remain fixed, then K achieves a maximum when $c = 85$. Find the sum of all distinct possible values of $a + b$.

10. Suppose that circles Ω_1 and Ω_2 intersect at P and Q, and that line AB is tangent to Ω_1 and Ω_2 at A and B, respectively, such that Q is closer to $\overline{AB}$ than P. If $AB = 2$, $PA = 20$, and $PB = 19$, then $QA \cdot QB$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

11. Adam, Bob, and Charlie each flip a coin every day, starting from Day 1, until all three of them have flipped heads at least once. The last of them to flip heads for the first time does so on Day X. The probability that X is even can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. There are nonzero real numbers x, y, z such that

$$\begin{aligned} 0 &= x^2y + 2x - y \\ &= y^2z + 2y - z \\ &= z^2x + 2z - x. \end{aligned}$$

Let T denote the least possible value of $100|xyz|$. Find the greatest integer that does not exceed T.

13. Circles Γ_1, Γ_2, and Γ_3 are pairwise externally tangent and have diameters of length 70, 99, and 55, respectively. Suppose that Γ_2 and Γ_3 touch at X, Γ_3 and Γ_1 touch at Y, and Γ_1 and Γ_2 touch at Z. A point A is chosen on the minor arc YZ of Γ_1. Ray AZ intersects Γ_2 again at B, ray BX intersects Γ_3 again at C, ray CY intersects Γ_1 again at D, ray DZ intersects Γ_2 again at E, ray EX intersects Γ_3 again at F, and ray FY intersects Γ_1 again at G. Find the maximum possible value of $AB + BC + CD + DE + EF + FG$.

14. Consider all positive integers k for which there exists a positive integer n such that

$$n^4 + \frac{n^3 + n^2}{2} + n + 1 = k^2.$$

Find the greatest of all such k.

15. In triangle ABC, $AB = 11$, $BC = 19$, and $CA = 20$. Let O denote the circumcenter of $\triangle ABC$, and D, E, and F denote the feet of the altitudes from A, B, and C, respectively. Points X and Y are the feet of the perpendiculars from E and F, respectively, to $\overline{AD}$. If $\overline{AO}$ intersects $\overline{EF}$ at Z, then there exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. Suppose that $\overline{ZT}$ intersects $\overline{AD}$ at P. Then, there exist relatively prime positive integers m and n such that $\frac{PX}{PY} = \frac{m}{n}$. Find $m + n$.