Black Group Tests

MOP 2023

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Contents

§1 Problems

§1.1 MOP Test 1

Problem K1.1 (ISL 2022 G3)**.** Let $ABCD$ be a cyclic quadrilateral, and let P and Q be points on line AB such that line AC is tangent to the circumcircle of $\triangle ADQ$ and line BD is tangent to the circumcircle of $\triangle BCP$. Let M and N be the midpoints of segments BC and AD respectively. Prove that the tangent to the circumcircle of $\triangle ANQ$ at A and the tangent to the circumcircle of $\triangle BMP$ at B intersect on line CD.

Problem K1.2 (ISL 2022 A4)**.** Determine the largest constant $c > 0$ such that, for any integer $n \geq 3$ and for any reals x_1, x_2, $\ldots$, x_n in $[0, 1]$ whose sum s is at least 3, there exist integers i and j with $1 \leq i < j \leq n$ and

$$2^{j-i}x_ix_j > c \cdot 2^s.$$

Problem K1.3 (ISL 2022 N6)**.** Prove that there exist a positive real c and a positive integer N_0 such that the following holds:

> Let Q be any set of prime numbers. For each positive integer n,
>
> - let $p(n)$ denote the number of primes dividing n, counted with multiplicty, and

- let $q(n)$ denote the number of primes in Q dividing n, counted with multiplicity.

Then, for any positive integer $N > N_0$, there exist at least cN positive integers n in $\{1, \ldots, N\}$ such that $p(n) + p(n+1)$ and $q(n) + q(n+1)$ are both even.

§1.2 MOP Test 2

Problem K2.1 (ISL 2022 C4). Let $n \geq 3$ be an integer. There are n coins distributed to n children in a circle. At every step, a child with at least two coins gives one coin to each of their neighbors. Determine all initial distributions of the coins such that it is possible for all children to have exactly one coin after a finite number of steps.

Problem K2.2 (ISL 2022 G6). Let ABC be an acute triangle and let H be the foot from A to $\overline{BC}$. Let P be a variable point such that the internal angle bisectors k and ℓ of $\angle PBC$ and $\angle PCB$, respectively, meet on $\overline{AH}$. Let k meet $\overline{AC}$ at E, ℓ meet $\overline{AB}$ at F, and $\overline{EF}$ and $\overline{AH}$ at Q. Prove that as P varies, line PQ passes through a fixed point.

Problem K2.3 (ISL 2022 A5). Find all integers $n \geq 2$ for which there exists real numbers $a_1 < \cdots < a_n$ such that the $\binom{n}{2}$ numbers of the form $a_j - a_i$ (for $1 \leq i < j \leq n$) can be rearranged to form a geometric progression.

§1.3 MOP Test 3

Problem K3.1 (ISL 2022 N2). Find all integers $n \geq 3$ such that $n!$ divides the product

$$\prod_{\substack{p<q\leq n \\ p,\, q \text{ prime}}} (p+q).$$

Problem K3.2 (ISL 2022 G4). Let ABC be an acute scalene triangle with circumcenter O, and let D be a point on side BC. The line through D perpendicular to $\overline{BC}$ meets lines AO, AC, and AB at W, X, and Y, respectively. The circumcircles of $\triangle AXY$ and $\triangle ABC$ intersect again at $Z \neq A$. Prove that if $W \neq D$ and $OW = OD$, then line DZ is tangent to the circumcircle of $\triangle AXY$.

Problem K3.3 (ISL 2022 C9). Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers, and let $f : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Z}_{\geq 0}$ be a bijection such that for any four nonnegative integers x_1, x_2, y_1, y_2 satisfying $f(x_1, y_1) > f(x_2, y_2)$, it holds that $f(x_1+1, y_1) > f(x_2+1, y_2)$ and $f(x_1, y_1+1) > f(x_2, y_2+1)$.

Let N be the number of integer pairs (x, y) with $0 \leq x, y < 100$ for which $f(x, y)$ is odd. As f varies, determine the smallest and largest possible value of N.

§1.4 MOP Test 4

Problem K4.1 (ISL 2022 N3). Let $a \geq 2$ and $d \geq 2$ be relatively prime integers. Let $x_1 = 1$ and for $k \geq 1$, define

$$x_{k+1} = \begin{cases} x_k + d & \text{if } a \text{ doesn't divide } x_k, \\ x_k/a & \text{if } n \text{ divides } x_k. \end{cases}$$

Find the greatest positive integer n for which some term of the sequence is divisible by a^n.

Problem K4.2 (ISL 2022 C6). At MOP, there are a finite number of students who are grouped into different classrooms. At every step, Po may remove an equal number of students from two classrooms and put them all in a new empty classroom. Determine, in terms of the initial grouping, the smalest possible number of nonempty classrooms he can obtain after a finite number of steps.

Problem K4.3 (ISL 2022 A7)**.** Let $s(m)$ denote the sum of the digits of a positive integer m. Determine whether there exists a polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, for some $n \geq 2$, such that

- a_0, a_1, …, a_{n-1} are positive integers, and
- for all positive integers k, $s(k) + s(P(k))$ is even.

§1.5 ELMO

Problem ELMO1 (Raymond Feng)**.** Let m be a positive integer. Find, in terms of m, all polynomials $P(x)$ with integer coefficients such that for every integer n, there exists an integer k such that $P(k) = n^m$.

Problem ELMO2 (Raymond Feng)**.** Let a, b, and n be positive integers. A lemonade stand owns n cups, all of which are initially empty. The lemonade stand has a *filling machine* and an *emptying machine*, which operate according to the following rules:

- If at any moment, a completely empty cups are available, the filling machine spends the next a minutes filling those a cups simultaneously and doing nothing else.
- If at any moment, b completely full cups are available, the emptying machine spends the next b minutes emptying those b cups simultaneously and doing nothing else.

Suppose that after a sufficiently long time has passed, both the filling machine and emptying machine work without pausing. Find, in terms of a and b, the least possible value of n.

Problem ELMO3 (Holden Mui)**.** Convex quadrilaterals $ABCD$, $A_1B_1C_1D_1$, and $A_2B_2C_2D_2$ are similar with vertices in order. Points A, A_1, B_2, B are collinear in order, points B, B_1, C_2, C are collinear in order, points C, C_1, D_2, D are collinear in order, and points D, D_1, A_2, A are collinear in order. Diagonals AC and BD intersect at P, diagonals A_1C_1 and B_1D_1 intersect at P_1, and diagonals A_2C_2 and B_2D_2 intersect at P_2. Prove that points P, P_1, and P_2 are collinear.

Problem ELMO4 (Luke Robitaille)**.** Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects $\overline{AC}$ at E and line CH intersects $\overline{AB}$ at F. Let X be the foot of the perpendicular from H to the line through A parallel to $\overline{EF}$. Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to $\overline{AC}$, and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to $\overline{AB}$. Prove that points B, C, B_1, C_1 are concyclic.

Problem ELMO5 (Karthik Vedula)**.** Find the least positive integer M for which there exist a positive integer n and polynomials $P_1(x)$, $P_2(x)$, …, $P_n(x)$ with integer coefficients satisfying

$$Mx = P_1(x)^3 + P_2(x)^3 + \cdots + P_n(x)^3.$$

Problem ELMO6 (Brandon Wang, Edward Wan)**.** For a set S of positive integers and a positive integer n, consider the game of (n, S)-*nim*, which is as follows. A pile starts with n watermelons. Two players, Deric and Erek, alternate turns eating watermelons from the pile, with Deric going first. On any turn, the number of watermelons eaten must be an element of S. The last player to move wins. Let $f(S)$ denote the set of positive integers n for which Deric has a winning strategy in (n, S)-nim.

Let T be a set of positive integers. Must the sequence

$$T,\ f(T),\ f(f(T)),\ \ldots$$

be eventually constant?

§1.6 Mock IMO

Problem MIMO1 (ISL 2022 A2). Let $k \geq 2$ be an integer. A nonempty set S of real numbers has the property that every element $s \in S$ can be written as the sum of k distinct elements of S that are not equal to s. Find the smallest possible value of $|S|$, in terms of k.

Problem MIMO2 (ISL 2022 A6). Find all rational numbers q for which there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + f(y)) = f(x) + f(y) \quad \text{and} \quad f(z) \neq qz$$

for all real numbers x, y, z.

Problem MIMO3 (ISL 2022 G8). Let $AA'BCC'B'$ be a convex cyclic hexagon such that line AC is tangent to the incircle of $\triangle A'B'C'$ and line $A'C'$ is tangent to the incircle of $\triangle ABC$. Let lines AB and $A'B'$ intersect at X and let lines BC and $B'C'$ intersect at Y. Prove that if $XBYB'$ is a convex quadrilateral, then it has an incircle.

Problem MIMO4 (ISL 2022 G2). Point P lies in the interior of acute triangle ABC such that lines AP and BC are perpendicular. Points D and E on side BC satisfy $\overline{PD} \parallel \overline{AC}$ and $\overline{PE} \parallel \overline{AB}$, and points $X \neq A$ and $Y \neq A$ lie on the circumcircles of $\triangle ABD$ and $\triangle ACE$, respectively, such that $DA = DX$ and $EA = EY$. Prove that points B, C, X, and Y are concyclic.

Problem MIMO5 (ISL 2022 N5). For each $1 \leq i \leq 9$ and positive integer T, let $d_i(T)$ denote the total number of times the digit i appears when all multiples of 2023 between 1 and T inclusive are written out in base 10. Prove that there are infinitely many positive integers T such that there are exactly two distinct values among $d_1(T)$, $d_2(T)$, …, $d_9(T)$.

Problem MIMO6 (ISL 2022 C7). Let s be a positive integer. Lucy and Lucky play the following game on a blackboard. Lucy initially writes s integer-valued 2023-tuples on the board. Lucky then gives Lucy an integer-valued 2023-tuple. Afterwards, Lucy can repeatedly take any two (not necessarily distinct) tuples $(v_1, \ldots, v_{2023})$ and $(w_1, \ldots, w_{2023})$ on the blackboard and writes the tuples

$$(v_1 + w_1, \ldots, v_{2023} + w_{2023}) \quad \text{and} \quad (\max(v_1, w_1), \ldots, \max(v_{2023}, w_{2023}))$$

on the board. Lucy wins if she can write Lucky's tuple on the board in a finite number of steps.

Determine the smallest value of s for which Lucy has a winning strategy.

§2 Solutions

§2.1 Solutions to MOP Test 1

K1.1 — ISL 2022 G3

Let $ABCD$ be a cyclic quadrilateral, and let P and Q be points on line AB such that line AC is tangent to the circumcircle of $\triangle ADQ$ and line BD is tangent to the circumcircle of $\triangle BCP$. Let M and N be the midpoints of segments BC and AD respectively. Prove that the tangent to the circumcircle of $\triangle ANQ$ at A and the tangent to the circumcircle of $\triangle BMP$ at B intersect on line CD.

Let the tangent to (ANQ) at A intersect (ADQ) at S. Then $\measuredangle DQS = \measuredangle NAS = \measuredangle NQA$ implies $\overline{QS}$ is a symmedian of $\triangle ANQ$, hence

$$-1 = (AD;QS) \stackrel{A}{=} (C, D; \overline{AB} \cap \overline{CD}, \overline{AS} \cap \overline{CD}).$$

Symmetrically, we conclude $\overline{AS}$ and the corresponding tangent at B both intersect at the harmonic conjugate of $\overline{AB} \cap \overline{CD}$ with respect to $\overline{CD}$.

K1.2 — ISL 2022 A4

Determine the largest constant $c > 0$ such that, for any integer $n \geq 3$ and for any reals $x_1, x_2, \ldots, x_n$ in $[0, 1]$ whose sum s is at least 3, there exist integers i and j with $1 \leq i < j \leq n$ and

$$2^{j-i} x_i x_j > c \cdot 2^s.$$

The answer is $c = \frac{1}{8}$.

Take i and j so that $2^{j-i}x_ix_j$ is maximal; then $2^{-i}x_i$ is maximal for all choices of $i < j$ and $2^j x_j$ is maximal for all choices of $j > i$. We will make changes without increasing $2^{j-i}x_ix_j \cdot 2^{-s}$.

Note that:

- We may always increase x_k for $k < i$ to $2^{k-i}x_i$, as this only increases s without affecting the optimality of the choice of (i, j), and thus does not increase $\max_{i,j} 2^{j-i}x_ix_j \cdot 2^{-s}$.
- Similarly, we may always increase x_k for $k > j$ to $2^{j-k}x_j$.
- Additionally, we may extend the sequence indefinitely in both directions, and increase such terms in accordance with the previous two bullet points.
- Similarly, we may always increase x_k for $i < k < j$ to $\min\{1, 2^{k-i}x_i, 2^{j-k}x_i\}$.
- While $j - i \geq 2$ and $x_i \leq \frac{1}{2}$, we have $x_{i+1} = 2x_i$ and hence we may replace i with $i + 1$ without changing the value of $2^{j-i}x_ix_j$.
- Similarly while $j - i \geq 2$ and $x_j \leq \frac{1}{2}$, we may replace j with $j - 1$.

Hence our sequence looks like (for large ℓ)

$$2^{-\ell}x_i,\ ,\ldots,\ 2^{-1}x_i,\ x_i,\ \underbrace{1,\ \ldots,\ 1}_{t \text{ ones}},\ x_j,\ 2^{-1}x_j,\ \ldots,\ 2^{-\ell}x_j.$$

Then we have $s \to 2x_i + 2x_j + t$ and $j - i = t + 1$, so

$$2^{j-i}x_ix_j \cdot 2^{-s} > \frac{2^{t+1}x_ix_j}{2^{2x_i+2x_j+t}} = 2 \cdot \frac{x_i}{4^{x_i}} \cdot \frac{x_j}{4^{x_j}} \geq 2 \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{8},$$

since $x_i, x_j \in [1/2, 1]$.

We achieve $c \to 1/8$ by taking $\ell \to \infty$ in the sequence

$$2^{-\ell},\ 2^{-\ell+1},\ \dots,\ 2^{-2},\ 2^{-1},\ 2^{0},\ 2^{-1},\ 2^{-2},\ \dots,\ 2^{-\ell+1},\ 2^{-\ell}.$$

K1.3 — ISL 2022 N6

Prove that there exist a positive real c and a positive integer N_0 such that the following holds:

> Let Q be any set of prime numbers. For each positive integer n,
>
> - let $p(n)$ denote the number of primes dividing n, counted with multiplicty, and
> - let $q(n)$ denote the number of primes in Q dividing n, counted with multiplicity.
>
> Then, for any positive integer $N > N_0$, there exist at least cN positive integers n in $\{1, \dots, N\}$ such that $p(n) + p(n+1)$ and $q(n) + q(n+1)$ are both even.

For each positive integer N, consider

$$S = \{5040N,\ 5040N + 70,\ 5040N + 72,\ 5040N + 75,\ 5040N + 80\}.$$

There are 4 possible values of $\{p(n) \bmod 2, q(n) \bmod 2\}$, so by Pigeonhole, there are distinct a and b in S with $p(a) \equiv p(b) \pmod 2$ and $q(a) \equiv q(b) \pmod 2$.

But S is constructed with the property that for any $a < b$ in S, $b - a$ divides both a and b. Hence

$$p\left(\frac{a}{b-a}\right) + p\left(\frac{b}{b-a}\right) = p(a) + p(b) - 2p(b-a) \equiv 0 \pmod 2,$$

and similarly

$$q\left(\frac{a}{b-a}\right) + q\left(\frac{b}{b-a}\right) \equiv 0 \pmod 2.$$

Since $\frac{a}{b-a}$ and $\frac{b}{b-a}$ are one apart, each N generates a valid n.

Hence for each M, the first M choices of N generate M different n up to $5040N + 80$. Each value of n is counted at most $\binom{5}{2}$ times, so any $c < \frac{1}{50400}$ works.

§2.2 Solutions to MOP Test 2

K2.1 — ISL 2022 C4

Let $n \geq 3$ be an integer. There are n coins distributed to n children in a circle. At every step, a child with at least two coins gives one coin to each of their neighbors. Determine all initial distributions of the coins such that it is possible for all children to have exactly one coin after a finite number of steps.

Label the children 1, 2, $\dots$, n and let them have a_1, a_2, $\dots$, a_n coins, respectively (with indices modulo n). Note that $a_1 + 2a_2 + \cdots + na_n$ is invariant modulo n, so for the desired distribution to be reachable, we must have

$$n \;\Bigg|\; \sum_{i=1}^{n} i(a_i - 1). \tag{$\star$}$$

We show that if the initial distribution satisfies the above, then the desired distribution is reachable.

Choose integers $d_1, \ldots, d_n$ such that

$$x_i - 1 = d_i - d_{i+1} \quad \text{for all } i.$$

With $(\star)$, we can check that $n \mid d_1 + \cdots + d_n$, so by shifting d_i appropriately, we may set $d_1 + \cdots + d_n = 0$.

Then we may choose integers $a_1, \ldots, a_n$ such that

$$d_i = a_i - a_{i-1} \quad \text{for all } i \quad \text{and} \quad \min_i a_i = 0.$$

Thus we have nonnegative integers $a_1, \ldots, a_n$ with

$$x_i - 1 = 2a_i - a_{i-1} - a_{i+1} \quad \text{for all } i.$$

While there exists i with $x_i \geq 2$, note that

$$1 \leq x_i - 1 \leq 2a_i \implies a_i \geq 1,$$

so decrease a_i by 1. This is equivalent to decreasing x_i by 2 and increasing x_{i-1} and x_{i+1} by 1. Do this until $x_i \leq 1$ for all i, and of course equality holds.

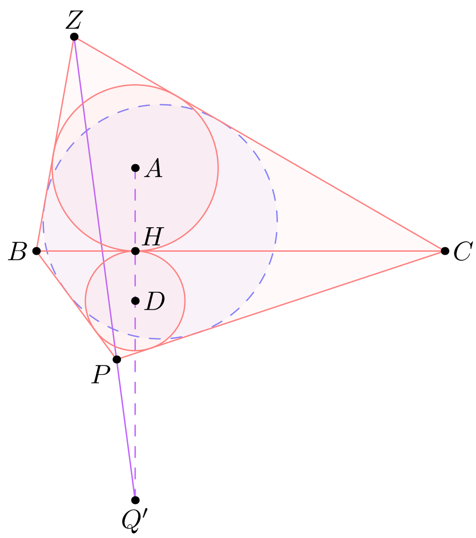
K2.2 — ISL 2022 G6

Let ABC be an acute triangle and let H be the foot from A to $\overline{BC}$. Let P be a variable point such that the internal angle bisectors k and ℓ of $\angle PBC$ and $\angle PCB$, respectively, meet on $\overline{AH}$. Let k meet $\overline{AC}$ at E, ℓ meet $\overline{AB}$ at F, and $\overline{EF}$ and $\overline{AH}$ at Q. Prove that as P varies, line PQ passes through a fixed point.

Let $D = k \cap \ell$, and let Z be the intersection of the reflection of $\overline{BC}$ over $\overline{AB}$ and the reflection of $\overline{BC}$ over $\overline{CA}$. Then A is an incenter or excenter of $\triangle ZBC$ and D is an incenter or excenter of $\triangle PBC$. We assume both are incenters without loss of generality (the other cases are analogous). We show Z is the fixed point.

If we define $Q' = \overline{ZP} \cap \overline{AD}$, we wish to show Q' is collinear with $E = \overline{AC} \cap \overline{DB}$ and $F = \overline{AB} \cap \overline{DC}$. By Ceva-Menelaus, we wish to show:

> Let $ZBPC$ be a quadrilateral. Let ω_A and ω_D be the incircles and A and D the incenters of $\triangle ZBC$ and $\triangle PBC$, respectively. If ω_A and ω_D are tangent at a point H on $\overline{BC}$, then $Q' = \overline{ZP} \cap \overline{AD}$ satisfies $(AD; HQ') = -1$.



Note that

$$\frac{ZB+BC-ZC}{2}=BH=\frac{PB+BC-PC}{2}\implies ZB-ZC=PB-PC,$$

so $ZBPC$ has an incircle ω by Pitot.

Note that H is the insimilicenter of ω_A and ω_D. By Monge on ω, ω_A, ω_D, the exsimilicenter is Q'. The desired harmonic bundle follows.

> **Remark.** Another proof is to draw the hyperbola $\mathcal{H}$ with foci B and C through Z, H, P, and note that $A=\overline{ZZ}\cap\overline{PP}$ and $D=\overline{HH}\cap\overline{PP}$. Then the desired harmonic bundle follows from duality.
>
> This also works as a standalone proof by initially setting $\mathcal{H}$ as the hyperbola with foci B and C through H and P, and identifying Z as the intersection of the second tangent from A to $\mathcal{H}$.

> **Remark.** The fixed point $Z=\overline{AO}\cap(OBC)$ may be identified by the following:
>
> - Setting $D\to A$ gives that the fixed point lies on $\overline{AO}$.
> - Setting D as the reflection of A over $\overline{BC}$ gives that the fixed point lies on the line through $\overline{AO}\cap(OBC)$ perpendicular to $\overline{BC}$.

> **Remark.** The problem is also not hard to coordinate bash. Perhaps this is slightly more difficult if the fixed point is actually provided in the problem statement.

K2.3 — ISL 2022 A5

Find all integers $n\geq 2$ for which there exists real numbers $a_1<\cdots<a_n$ such that the $\binom{n}{2}$ numbers of the form a_j-a_i (for $1\leq i<j\leq n$) can be rearranged to form a geometric progression.

The answer is $n\leq 4$, constructed as follows:

- For $n=2$, take $a_2-a_1=1$.
- For $n=3$, take $a_2-a_1=1$ and $a_3-a_2=x$, where $x^2=x+1$.
- For $n=4$, take $a_2-a_1=1$, $a_3-a_2=x$, and $a_4-a_3=x^2$, where $x^3=x+1$.

It suffices to show $n\leq 4$ is necessary.

By scaling, let the $\binom{n}{2}$ differences be 1, x, x^2, …, $x^{\binom{n}{2}-1}$, where $x>1$. For $0\leq k\leq\binom{n}{2}-1$, let $I_k=[L_k,R_k]$ such that $a_{R_k}-a_{L_k}=x^k$, and let $d_k=R_k-L_k$.

> **Claim 1.** I_k and I_{k+1} share at least a point for each k.

Proof. Assume for contradiction there is an interval I_ℓ of positive length between the closest endpoints of I_k and I_{k+1}, and let $I_u=I_k\cup I_\ell$ and $I_v=I_{k+1}\cup I_\ell$.

Of course $v>u>k+1$, but

$$x^{k+1}-x^k=x^v-x^u\geq x^{u+1}-x^u>x^{k+1}-x^k,$$

contradiction. □

Of course $d_0=d_1=1$. Let r be minimal such that $d_r>1$.

Claim 2. $r \le 3$ and $x^r = x + 1$.

Proof. Since I_k and I_{k+1} share a point for each k and $d_k = 1$ for $k < r$, the intervals I_0, I_1, …, I_{r-1} appear in order.

The smallest interval of length greater than 1 is $I_0 \cup I_1$, so $I_r = I_0 \cup I_1$, implying $x^r = x + 1$. Moreover, we readily have $I_{r+1} = I_1 \cup I_2$, …, $I_{2r-2} = I_{r-2} \cup I_{r-1}$.

Therefore if $I_s = I_0 \cup I_1 \cup I_2$, then $s \ge 2r - 1$, so we have

$$1 + x + x^2 = x^s \ge x^{2r-1}.$$

In particular,

$$x + x^2 + x^3 \ge x^{2r} = (x+1)^2 \implies x^3 \ge x + 1 = x^r,$$

so $r \le 3$. $\square$

Claim 3. $n = r + 1$.

Proof. As previously established, the intervals I_0, I_1, …, I_{r-1} appear in order, so $n \ge r + 1$.

Let $R = \binom{r+1}{2}$. We may check in both cases $r = 2$ and $r = 3$ the R differences induced by the $r+1$ endpoints of I_0, …, I_{r-1} induce the $\binom{R}{2}$ differences 1, x, x^2, …, x^{R-1}. (In particular, $I_{R-1} = I_0 \cup I_1 \cup \cdots \cup I_{r-1}$.)

If $n \ge r+2$, then it is necessary to locate I_R, which must intersect I_{R-1}. But if I_{R-1} intersects I_R at an interval I_t of positive length, then $x^R - x^t$, the length of the interval $I_R \setminus I_t$, must be a power of x less than x^R. But we have identified all intervals of such length, contradiction.

Thus I_R intersects I_{R-1} at an endpoint. But then:

- If I_R intersects I_{R-1} at an endpoint in I_0, then the intervals $I_R \cup I_0$, $I_R \cup I_0 \cup I_1$, …, $I_R \cup I_0 \cup \cdots \cup I_{r-2}$ must all have length strictly between x^R and $x^R + x^{R-1} = x^{R-1+r}$, but there are only $r - 2$ distinct possible interval lengths strictly between x^R and x^{R-1+r}, contradiction.
- If I_R intersects I_{R-1} at an endpoint in I_{r-1}, we derive a similar contradiction with the intervals $I_R \cap I_{r-1}$, …. $\square$

Since $r \le 3$, we have $n \le 4$.

§2.3 Solutions to MOP Test 3

K3.1 — ISL 2022 N2

Find all integers $n \ge 3$ such that $n!$ divides the product

$$\prod_{\substack{p<q\le n \\ p,\, q \text{ prime}}} (p+q).$$

The answer is $n = 7$, which works. We may manually check all other $n \le 10$ fail, and proceed to show $n \ge 11$ fail as well.

Let $r \ge 11$ be the largest prime at most n. It suffices to show

$$r! \quad \text{does not divide} \quad \prod_{p<q\le r} (p+q).$$

Assume for contradiction $r!$ does divide the product.

Claim 1. $r-2$ is prime.

Proof. Since $p+q<2r$, the only way for a factor of r to appear in the product is if $p+q=r$, forcing $(p,q)=(2,r-2)$. Hence $r-2$ must be prime. □

Claim 2. $r-4$ is prime.

Proof. Since $p+q<3(r-2)$, the only ways for a factor of $r-2$ to appear in the product is if $p+q=r-2$ or $p+q=2(r-2)$.

- In the former case, we must have $(p,q)=(2,r-4)$, so $r-4$ is prime.
- In the latter case, note if $q \le r-2$ then $p+q<2(r-2)$, so we must have $(p,q)=(r-4,r)$. In particular, $r-4$ is prime. □

But for $r \ge 11$, it is absurd for r, $r-2$, $r-4$ to be prime.

K3.2 — ISL 2022 G4

Let ABC be an acute scalene triangle with circumcenter O, and let D be a point on side BC. The line through D perpendicular to $\overline{BC}$ meets lines AO, AC, and AB at W, X, and Y, respectively. The circumcircles of $\triangle AXY$ and $\triangle ABC$ intersect again at $Z \neq A$. Prove that if $W \neq D$ and $OW = OD$, then line DZ is tangent to the circumcircle of $\triangle AXY$.

Let A', B', C' be the antipodes of A, B, C, so W lies on $\overline{B'C'}$. Let Z' and H_A lie on (ABC) with $\overline{AH_A} \perp \overline{BC}$ and $\overline{AZ} \parallel \overline{BC}$.

By Pascal on $H_AZ'C'BCA$, the line through D perpendicular to $\overline{BC}$ contains $\overline{Z'C'} \cap \overline{CA}$, so $X \in \overline{Z'C'}$. Similarly $Y \in \overline{Z'B'}$. Moreover

$$\measuredangle YZ'X = \measuredangle B'ZC' = \measuredangle BAC = \measuredangle YAX,$$

so $Z' \in (AXY)$ and thus $Z = Z'$.

Finally Z is the Miquel point of $BCXY$, so Z also lies on (YBD) and (XCD). Then

$$\measuredangle DZX = \measuredangle DCX = \measuredangle ZAC = \measuredangle ZYX,$$

implying $\overline{DZ}$ is tangent to (XYZ), as desired.

K3.3 — ISL 2022 C9

Let $\mathbb{Z}_{\ge 0}$ denote the set of nonnegative integers, and let $f : \mathbb{Z}_{\ge 0}^2 \to \mathbb{Z}_{\ge 0}$ be a bijection such that for any four nonnegative integers x_1, x_2, y_1, y_2 satisfying $f(x_1,y_1) > f(x_2,y_2)$, it holds that $f(x_1+1,y_1) > f(x_2+1,y_2)$ and $f(x_1,y_1+1) > f(x_2,y_2+1)$.

Let N be the number of integer pairs (x,y) with $0 \le x, y < 100$ for which $f(x,y)$ is odd. As f varies, determine the smallest and largest possible value of N.

The minimum and maximum values are 2500 and 7500 respectively.

Proof of bounds: Note that $f(x,y) < f(x+1,y)$ for all x and y, else

$$f(x,y) > f(x+1,y) > f(x+2,y) > \cdots$$

which is absurd. Similarly $f(x, y) < f(x, y+1)$.

Consider the process of placing the elements of $\mathbb{Z}_{\geq 0}$ in the cells of $\mathbb{Z}^2_{\geq 0}$ in the order 0, 1, 2, …, so that n is placed in cell (x, y) when $f(x, y) = n$. In this process, when n is placed in (x, y), then the cells $(x-1, y)$ and $(x, y-1)$ must already have been labeled, or they will be filled with a smaller number, contradicting the above. Hence the set of labeled cells will always form a "chomp-like" structure.

Claim 3. Consider the moment cell (x, y) is filled with $f(x, y) = n$, and let $f(x, y-1) = m$. In every other column with a positive number of labeled cells, exactly one cell was labeled between when m and n were labeled.

Proof. We simply check that:

- If some nonempty column had no cell filled during this time, and its top cell is $f(x', y') < m$, then the cell $(x', y'+1)$ must be filled eventually, so $f(x', y'+1) > n$. This is a clear contradiction.

- If some nonempty column was filled twice, then its top two cells must satisfy $m < f(x', y') < n$ and $m < f(x', y'+1) < n$. This is again a clear contradiction. □

Analogously, we have the reflected version of the above claim.

Claim 4. We always have

$$f(x, y) + f(x+1, y+1) = f(x+1, y) + f(x, y+1) + 1.$$

Proof. Without loss of generality $f(x, y+1) > f(x+1, y)$. By Claim 1,

$$\begin{aligned} f(x, y+1) - f(x, y) &= \#\text{ nonempty columns when } f(x, y+1) \text{ placed} \\ \text{and} \quad f(x+1, y+1) - f(x+1, y) &= \#\text{ nonempty columns when } f(x+1, y+1) \text{ placed.} \end{aligned}$$

Hence it suffices to show that between when $f(x, y+1)$ and $f(x+1, y+1)$ are placed, the number of nonempty columns increases by exactly once.

But by the reflected version of Claim 1, we note that the bottom row increases by exactly one square between when $f(x, y+1)$ and $f(x+1, y+1)$ are placed. Since the cells form a chomp-like structure, this proves the claim. □

Hence in any 2×2 square, there is either 1 or 3 odd terms. This readily establishes the bounds of 2500 and 7500.

Constructions: We show:

Claim 5. Where $r < n$ below, the function

$$f(qn + r, y) = [1 + 2 + \cdots + (q+y)]n + q + (q+y+1)r$$

works.

Proof. It is clear f is a bijection: for each m, we may find a unique value of $s \geq 0$ with

$$m = (1 + 2 + \cdots + s)n + R, \quad \text{where } R < (s+1)n,$$

and pick $q \equiv R \pmod{s+1}$ with $0 \leq q \leq s$, $y = s - q$, and $r = \lfloor \frac{R}{s+1} \rfloor < n$.

Then we may check that

$$\begin{aligned} f(qn+r,y+1)-f(qn+r,y) &= (q+y+1)n+r \\ &= (s+1)n+\left\lfloor \frac{R}{s+1} \right\rfloor \\ \text{and}\quad f(qn+r+1,y)-f(qn+r,y) &= \begin{cases} q+y+1 & \text{if } r \le n-2 \\ q+y+2 & \text{if } r = n-1 \end{cases} \\ &= \begin{cases} s+1 & \text{if } R < (n-1)(s+1) \\ s+2 & \text{if } R \ge (n-1)(s+1) \end{cases} \end{aligned}$$

are increasing functions in m, which is sufficiently to verify the requested condition on f. □

Claim 6. Where $r < n$ below, the function

$$f(qn+r,y) = [1+2+\cdots+(q+y)]n+y+(q+y+1)r$$

works.

Proof. The proof is analogous to the above, where instead we verify

$$\begin{aligned} f(qn+r,y+1)-f(qn+r,y) &= (q+y+1)n+r+1 \\ &= (s+1)n+\left\lfloor \frac{R}{n+1} \right\rfloor + 1 \\ \text{and}\quad f(qn+r+1,y)-f(qn+r,y) &= q+y+1 = s+1 \end{aligned}$$

are increasing. □

Now taking $n \gg 100$ even in Claim 3 gives $f(x,y) \equiv x(y+1) \pmod 2$ and $n \gg 100$ even in Claim 4 gives $f(x,y) \equiv (x-1)(y-1)-1 \pmod 2$. The former is odd 2500 times and the latter is odd 7500 times.

Remark. The above solution is entirely combinatorial, in contrast to the more analytical official solution, which analyzes all vectors (k,ℓ) based on whether $f(x,y) < f(x+k,y+\ell)$ or $f(x,y) > f(x+k,y+\ell)$ and finds that the vectors of the former and latter types are divided by a single slope. It then comes to the following characterization of f:

> For some α, f sorts (x,y) based on the value of $x+y\alpha$. (For α rational, tiebreaking is done by larger x- or y-coordinate, depending on the nature of f.)

We also have the more explicit expression

$$f(x,y) = xy + \sum_{i=1}^{a} \left\lceil \frac{i}{\alpha} \right\rceil + \sum_{j=1}^{b} \lceil j\alpha \rceil.$$

Claim 2 follows from this and thus the bound. We may construct 2500 by setting $\alpha \approx 199.999$ and 7500 by setting $\alpha \approx 200.001$.

§2.4 Solutions to MOP Test 4

K4.1 — ISL 2022 N3

Let $a \geq 2$ and $d \geq 2$ be relatively prime integers. Let $x_1 = 1$ and for $k \geq 1$, define

$$x_{k+1} = \begin{cases} x_k + d & \text{if } a \text{ doesn't divide } x_k, \\ x_k/a & \text{if } n \text{ divides } x_k. \end{cases}$$

Find the greatest positive integer n for which some term of the sequence is divisible by a^n.

The answer is $\lfloor \log_a(ap) \rfloor$.

Claim 1. We have $x_k < ad$ for all k.

Proof. Starting from a number less than d, we can increase by d at most $a - 1$ times before encountering a multiple of a. This multiple of a is less than ad, and after dividing by a, we end up less than d again. □

Of course the above claim establishes the upper bound.

Claim 2. We have $x_k = 1$ for some $k > 1$.

Proof. The function $f : \{1, 2, \ldots, d-1\} \to \{1, 2, \ldots, d-1\}$ defined by

$$f(n) = \frac{n + \ell d}{a} \in \mathbb{Z} \quad \text{where } \ell \geq 0 \text{ minimal}$$

is a bijection with inverse $f^{-1}(m) = am \bmod d$. Hence $f^k(1) = 1$ for some k.

Evidently for each i, there is a $j > i$ with $f(x_i) = x_j$. Hence there is a $k > 1$ with $x_k = 1$. □

Let the elements before 1 in the sequence be

$$\ldots,\ a^r - d,\ a^r,\ a^{r-1},\ \ldots,\ 1.$$

Then we must have $a^r \in (d, ad)$, which uniquely determines a^r. The problem follows.

K4.2 — ISL 2022 C6

At MOP, there are a finite number of students who are grouped into different classrooms. At every step, Po may remove an equal number of students from two classrooms and put them all in a new empty classroom. Determine, in terms of the initial grouping, the smalest possible number of nonempty classrooms he can obtain after a finite number of steps.

Let n be the total number of students and let t be the greatest odd divisor of n. The answer is 1 if t divides the number of students in each class and 2 otherwise.

> **Remark.** The intended solution goes along the lines of splitting the classes into groups of size powers of two, and then combining them. However, the below solution still works and is stronger.

We cite the following problem:

Lemma (ISL 1994 C3)

Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts.

While there are at least three classrooms, Po can take any three of them and rearrange them into two classrooms. Thus Po can arrange all the students into two classrooms. It suffices to show that arranging the students into one classroom is possible if and only if t divides all the class sizes.

Proof of necessity: If initially there is a class whose size is not divisible by t, then there will always be a class whose size is not divisible by t. Thus we cannot ever have a single class of size n.

Proof of sufficiency: Divide all sizes by t, so we may assume $n = 2^k$ is a power of 2. As above, arrange the students into two classes. Then if the classes have positive size a and b, we must have $\nu_2(a) = \nu_2(b)$.

Then the operation $(a, b) \mapsto (2a, b - a)$ increments $\nu_2(a) = \nu_2(b)$ until it no longer less than k, at which point $(a, b) = (0, 2^k)$ and we are done.

K4.3 — ISL 2022 A7

Let $s(m)$ denote the sum of the digits of a positive integer m. Determine whether there exists a polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, for some $n \ge 2$, such that

- $a_0, a_1, \ldots, a_{n-1}$ are positive integers, and
- for all positive integers k, $s(k) + s(P(k))$ is even.

The answer is no. We claim:

Claim. There are large positive integers C and D such $C \gg D \gg 0$ such that if

$$Q(x) = Cx^4 + Cx^3 - Dx^2 + Cx + C,$$

then the polynomial $P(Q(x))$ has all coefficients positive.

Proof. Note that $(x^4 + x^3 + x + 1)^n$ has terms of all degrees $0, 1, \ldots, x^{4n}$ with coefficients at least 1.

Then the modification $(x^4 + x^3 - \varepsilon x^2 + x + 1)^n$ decreases coefficients by a polynomial amount in ε which tends to 0 as $\varepsilon \to 0^+$, so there is a sufficiently small choice of ε for which the signs of the coefficients of $(x^4 + x^3 + x + 1)^n$ are not affected.

Then for large C and $D \approx C\varepsilon$, we may set $Q(x) = Cx^4 + Cx^3 - Dx^2 + Cx + 1$, and the $Q(x)^n$ term dominates. $\square$

Then for r greater than the number of digits in all the coefficients of Q and $P \circ Q$, we may find constants a and b such that

$$s(Q(10^r)) = 9r + a \quad \text{and} \quad s(P(Q(10^r))) = b$$

since incrementing r creates an additional digit of 9 in $Q(10^r)$ generated by the x^2 term and no additional nonzero digits in $P(Q(10^r))$. Then one of $k = 10^r$ and $k = 10^{r+1}$ must fail.

Remark. Derek Liu claims an approach where you take massive $J \equiv 25 \pmod{100}$ such that $a^{n-1}J^{n-1}$ has a leading digit of 6, and pick appropriate r such that in $P(J \cdot 10^r)$, the $a_k J^k$ terms are roughly concatenated. If we repick r such that the $a^n J^n$ and $a^{n-1}J^{n-1}$ terms intersect at one digit, then $2 + 5 + 6$ turns into $3 + 1$, with different parity.

§2.5 Solutions to ELMO

ELMO1 — ELMO 2023/1

Let m be a positive integer. Find, in terms of m, all polynomials $P(x)$ with integer coefficients such that for every integer n, there exists an integer k such that $P(k) = n^m$.

For any positive $d \mid n$, the polynomials $P(x) = (x + a)^d$ and $P(x) = (-x + a)^d$ work. We show they are the only solutions.

First, there exists k_0 so that $P(k_0) = 0^m$. Instead consider the polynomial $Q(x) = P(x - k_0)$, which has 0 as a root and but still contains every mth power in its range. We will show that $Q(x)$ is of the form either x^d or $(-x)^d$ for some $d \mid n$.

For each prime p, there is some k_p with $Q(k_p) = p^m$. Since 0 is a root of Q, we have $x \mid Q(x)$ for every x. In particular, $k_p \mid P(k_p) = p^m$. This implies that

$$k_p \in \{1, p, p^2, \dots, p^m, -1, -p, -p^2, \dots, -p^m\} \quad \text{for all } p.$$

By the Pigeonhole principle, one of the following must occur:

- There is some $0 \le r \le m$ such that $k_p = p^r$ for infinitely many p. In particular, $Q(x^r) = x^m$ infinitely often, implying it holds for all real x, so $Q(x) = x^{m/r}$ for all x.
- There is some $0 \le r \le m$ such that $k_p = -p^r$ for infinitely many p. In particular, $Q(-x^r) = x^m$ infinitely often, implying it holds for all real x, so $Q(x) = (-x)^{m/r}$ for all x.

This completes the proof.

ELMO2 — ELMO 2023/2

Let a, b, and n be positive integers. A lemonade stand owns n cups, all of which are initially empty. The lemonade stand has a *filling machine* and an *emptying machine*, which operate according to the following rules:

- If at any moment, a completely empty cups are available, the filling machine spends the next a minutes filling those a cups simultaneously and doing nothing else.
- If at any moment, b completely full cups are available, the emptying machine spends the next b minutes emptying those b cups simultaneously and doing nothing else.

Suppose that after a sufficiently long time has passed, both the filling machine and emptying machine work without pausing. Find, in terms of a and b, the least possible value of n.

The answer is $2(a + b - \gcd(a, b))$. We view the problem through two models:

- the *discrete model*, where cups are filled instantly at the end of each a-minute period, and cups are emptied instantly at the end of each b-minute period; and
- the *continuous model*, where cups are filled at a constant rate during each a-minute period, and cups are emptied at a constant rate during each b-minute period.

We begin by assuming $\gcd(a, b) = 1$.

Lower bound for $\gcd(a, b) = 1$: Assume that at some time, say $t = 0$, both the filling machine and the emptying machine are starting their next cycle. Suppose that c cups are filled at $t = 0$.

Using the discrete model, it suffices to consider when t is a multiple of a or b.

- At $t = ka$, the number of full cups is $c + (ka \bmod b)$, whose maximum value is $c + b - 1$. For the machines to continue working without pausing, we must have $n \geq (c + b - 1) + a$.
- At $t = \ell b$, the number of full cups is $c - (\ell b \bmod a)$, whose minimum value is $c - a + 1$. For the machines to continue working without pausing, we must have $0 \leq (c - a + 1) - b$.

Thus $n \geq 2(a + b - 1)$.

Upper bound for $\gcd(a, b) = 1$**:** Assume $n = 2(a + b - 1)$, and consider the continuous model. Let t be the time and L be the total amount of liquid in the cups.

- When t is an integer and $L \geq a + b - 1$, there is at most $a - 1$ total liquid in (at most a) cups being filled and thus at least b totally filled cups. Hence the emptying machine is active and decreases L by 1 per minute.
- When t is an integer and $L \leq a + b - 1$, there is at least 1 total liquid (i.e. at most $b - 1$ empty space) in (at most b) cups being emptied and thus at least a totally empty cups. Hence the filling machine is active and increases L by 1 per minute.

Each minute, either both machines are active, or L gets 1 closer closer to $a + b - 1$ (if it is not equal to $a + b - 1$ already). The latter can only occur finitely many times, so L is eventually constant.

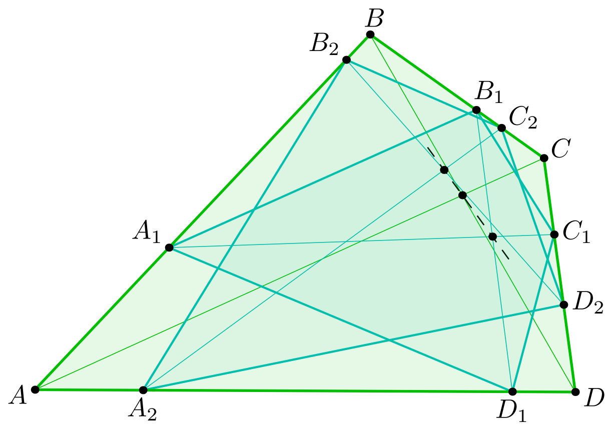
Finish for $\gcd(a, b) > 1$**:** From the perspective of the discrete model, events only happen when time is a multiple of $\gcd(a, b)$, and moreover the amount of total liquid is always a multiple of $\gcd(a, b)$.

Hence the problem for (a, b) is the problem for $(a/\gcd(a, b), b/\gcd(a, b))$, with time and liquid scaled up by $\gcd(a, b)$. It readily follows that the general answer is $2(a + b - \gcd(a, b))$.

ELMO3 — ELMO 2023/3

Convex quadrilaterals $ABCD$, $A_1B_1C_1D_1$, and $A_2B_2C_2D_2$ are similar with vertices in order. Points A, A_1, B_2, B are collinear in order, points B, B_1, C_2, C are collinear in order, points C, C_1, D_2, D are collinear in order, and points D, D_1, A_2, A are collinear in order. Diagonals AC and BD intersect at P, diagonals A_1C_1 and B_1D_1 intersect at P_1, and diagonals A_2C_2 and B_2D_2 intersect at P_2. Prove that points P, P_1, and P_2 are collinear.

Let X be the center of spiral similarity between $ABCD$ and $A_1B_1C_1D_1$, and let Y be the center of spiral similarity between $ABCD$ and $A_2B_2C_2D_2$. Let $\theta := \measuredangle XAB = \measuredangle XBC = \measuredangle XCD = \measuredangle XDA$ and $\theta' := \measuredangle ABY = \measuredangle BCY = \measuredangle CDY = \measuredangle DAY$.



Claim 1. $\theta = \theta'$; i.e. X and Y are isogonal conjugates.

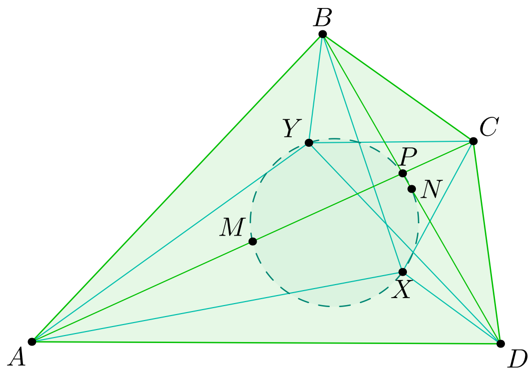
Proof. Assume for contradiction $\theta < \theta'$ (without loss of generality). Then by the law of sines in $\triangle XDA$ and $\triangle YAB$, we have

$$\frac{XD}{XA} = \frac{\sin(\angle A - \theta)}{\sin\theta} > \frac{\sin(\angle A - \theta')}{\sin\theta'} = \frac{YB}{YA}.$$

Multiplying cyclically gives

$$1 = \prod_{\text{cyc}} \frac{XD}{XA} > \prod_{\text{cyc}} \frac{YB}{YA} = 1,$$

contradiction. □



Claim 2. $ABCD$ is cyclic.

Proof. Note

$$\measuredangle AXD = \measuredangle XAD + \measuredangle ADX = \measuredangle XAD + \measuredangle BAX = \measuredangle BAD$$

and similarly $\measuredangle BXC = \measuredangle BCD$.

Since X has an isogonal conjugate,

$$\measuredangle BAD = \measuredangle AXD = \measuredangle BXC = \measuredangle BCD.$$ □

Claim 3. $ABCD$ is harmonic.

Proof. Observe that

$$\triangle AXD \sim \triangle AYB \implies \frac{YA}{XA} = \frac{AB}{AD}$$

$$\triangle BXA \sim \triangle BYC \implies \frac{YC}{XA} = \frac{BC}{AB}$$

$$\triangle CXB \sim \triangle CYD \implies \frac{YC}{XC} = \frac{CD}{BC}$$

$$\triangle DXC \sim \triangle DYA \implies \frac{YA}{XC} = \frac{AD}{BC}.$$

Hence

$$1 = \frac{YA}{XA} \cdot \frac{XC}{YA} \cdot \frac{YC}{XC} \cdot \frac{XA}{YC} = \frac{AB}{AD} \cdot \frac{BC}{AD} \cdot \frac{CD}{BC} \cdot \frac{AB}{BC},$$

implying $AB \cdot CD = AD \cdot BC$. □

Now let $P = \overline{AC} \cap \overline{BD}$, $P_1 = \overline{A_1C_1} \cap \overline{B_1D_1}$, $P_2 = \overline{A_2C_2} \cap \overline{B_2D_2}$. Let M be the midpoint of $\overline{AC}$ and N the midpoint of $\overline{BD}$.

Claim 4. X lies on (ABN) and (CDN); similarly Y lies on (ADN) and (BCN).

Proof. The first follows from $\measuredangle ANB = \measuredangle ADC = \measuredangle AXB$, and the rest follow analogously. $\square$

Claim 5. P, X, Y, M, N are concyclic.

Proof. We have

$$\measuredangle XNP = \measuredangle XAB = \measuredangle XBC = \measuredangle XMP,$$

implying $X \in (PMN)$, and similarly $Y \in (PMN)$. $\square$

Finally

$$\measuredangle XPY = \measuredangle XNY = \measuredangle XNA + \measuredangle ANY = \measuredangle XBA + \measuredangle ADY = 2\theta,$$

so

$$\measuredangle P_1PP_2 = \measuredangle P_1PX + \measuredangle XPY + \measuredangle YPP_2 = (-\theta) + 2\theta + (-\theta) = 0^\circ,$$

as desired.

ELMO4 — ELMO 2023/4

Let ABC be an acute scalene triangle with orthocenter H. Line BH intersects $\overline{AC}$ at E and line CH intersects $\overline{AB}$ at F. Let X be the foot of the perpendicular from H to the line through A parallel to $\overline{EF}$. Point B_1 lies on line XF such that $\overline{BB_1}$ is parallel to $\overline{AC}$, and point C_1 lies on line XE such that $\overline{CC_1}$ is parallel to $\overline{AB}$. Prove that points B, C, B_1, C_1 are concyclic.

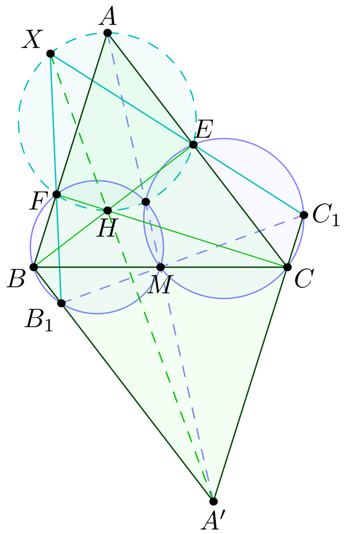
We present a few solutions. In each, let $A' = \overline{BB_1} \cap \overline{CC_1}$, so $ABA'C$ is a parallelogram.

First solution (author) Since $\overline{A'H} \perp \overline{EF}$, we have X, H, A' collinear. But

$$\measuredangle B_1XA' = \measuredangle FEH = \measuredangle FCB = \measuredangle XA'B_1,$$

implying $B_1X = B_1A'$. Similarly $C_1X = C_1A'$, so $\overline{B_1C_1} \perp \overline{XHA'}$.

This means $\overline{BC}$ and $\overline{B_1C_1}$ are antiparallel in $\angle A'$, so BB_1CC_1 is indeed cyclic.



Second solution (mine) Let M be the midpoint of $\overline{BC}$. Since $AEFX$ is an isosceles trapezoid and $ME = MF$,

$$\measuredangle B_1FM = \measuredangle XFM = \measuredangle MEA = \measuredangle ECM = \measuredangle B_1BM,$$

so $B_1 \in (BMF)$. Similarly $C_1 \in (CME)$.

But since $AB \cdot AF = AC \cdot AE$, line AM is the radical axis of (BMF) and (CME). In particular, A' lies on this radical axis, so $A'B \cdot A'B_1 = A'C \cdot A'C_1$ as needed.

Third solution (author) Let M be the midpoint of $\overline{BC}$.

Let ℓ be the perpendicular bisector of $\overline{EF}$ (so $M \in \ell$). Let B_2 is the reflection of B_1 in ℓ and let $M' \in \ell$ be the midpoint of $\overline{B_1B_2}$. Since $\overline{XF}$ and $\overline{AE}$ are reflections in ℓ, we know B_2 lies on $\overline{AC}$. If $M \neq M'$, this implies $\ell = \overline{MM'} \parallel \overline{AC}$, which is absurd. Hence M is the midpoint of $\overline{B_1B_2}$, i.e. $\overline{B_1M} \perp \ell$. Similarly $\overline{C_1M} \perp \ell$.

Then $\overline{B_1C_1} \parallel \overline{EF}$, implying $\overline{BC}$ and $\overline{B_1C_1}$ are antiparallel in $\angle A'$, which gives the desired.

ELMO5 — ELMO 2023/5

Find the least positive integer M for which there exist a positive integer n and polynomials $P_1(x)$, $P_2(x)$, …, $P_n(x)$ with integer coefficients satisfying

$$Mx = P_1(x)^3 + P_2(x)^3 + \cdots + P_n(x)^3.$$

The minimum value of M is 6, achieved by

$$6x = (x+1)^3 + (x-1)^3 + (-x)^3 + (-x)^3.$$

For the lower bound, write

$$Mx = \sum_{k=1}^{n} P_k(x)^3.$$

We will show $6 \mid M$, which suffices.

In $\mathbb{Z}[x]/(x^2+x+1)$, there are integers a_k and b_k for each k such that $P_k(x) = a_kx + b_k$. Then

$$\begin{aligned} Mx &= \sum_{k=1}^{n} (a_kx + b_k)^3 \\ &= \sum_{k=1}^{n} \left[a_k^3 + b_k^3 + 3a_k^2b_kx + 3a_kb_k^2(-x-1)\right] \\ &= \sum_{k=1}^{n} \left[a_k^3 + b_k^3 - 3a_kb_k^2 + 3a_kb_k(a_k - b_k)x\right], \end{aligned}$$

so we must have

$$M = \sum_{k=1}^{n} 3a_kb_k(a_k - b_k).$$

But $3a_kb_k(a_k - b_k)$ is a multiple of 6 for all integers a_k and b_k, so $6 \mid M$.

> **Remark.** Equivalently, one may substitute a primitive third root of unity for x.

ELMO6 — ELMO 2023/6

For a set S of positive integers and a positive integer n, consider the game of (n, S)-*nim*, which is as follows. A pile starts with n watermelons. Two players, Deric and Erek, alternate turns eating watermelons from the pile, with Deric going first. On any turn, the number of watermelons eaten must be an element of S. The last player to move wins. Let $f(S)$ denote the set of positive integers n for which Deric has a winning strategy in (n, S)-nim.

Let T be a set of positive integers. Must the sequence

$$T,\ f(T),\ f(f(T)),\ \dots$$

be eventually constant?

Yes, the sequence must be eventually constant. In what follows, let $\overline{S} = \mathbb{Z}_{\geq 0} \setminus S$, so $f(S) = S + \overline{f(S)}$ for all S. Note that $S \subseteq f(S)$ always, so the limit $f^\infty(T)$ is well-defined.

Claim 1. Let m be the smallest positive integer in $\overline{T}$. If all multiples of m are in $\overline{f^\infty(T)}$, then $f(T) = f^\infty(T)$.

Proof. Since $T \subseteq f^\infty(T)$, this means all multiples of m are in $\overline{T}$. However, $1, 2, \dots, m-1$ are in T. We may then compute that $f(T)$ is exactly the set of non-multiples of m, implying $f(T) = f^\infty(T)$. $\square$

Claim 2. If there are a and b with $a, b \in \overline{f^\infty(T)}$ but $a+b \in f^\infty(T)$, then $f^{j+3}(T) = f^\infty(T)$ for some j (defined below).

Proof. We take j as the index for which $n \in f^j(T) \iff n \in f^\infty(T)$ for $n \leq a+b$ (which must exist since $S \subseteq f(S)$ always).

For $n > a+b$, note:

- If $n-a-b \notin f^{j+1}(T)$ then $n = (a+b) + (n-a-b) \in f^{j+1}(T)$.
- If $n-a-b \in f^{j+1}(T)$, then $n-b = (n-a-b) + a \in f^{j+2}(T)$ so $(n-b)+b \in f^{j+3}(T)$.

Thus all $n > a+b$ are in $f^{j+3}(T) \subseteq f^\infty(T)$, implying $f^{j+3}(T) = f^\infty(T)$. $\square$

If $\overline{T} = \{0\}$ then we are already done. Otherwise m exists, we may check that $m \in \overline{f^\infty(T)}$. Then:

- If all multiples of m are in $\overline{f^\infty(T)}$, then Claim 1 finishes.
- Otherwise, let ℓm be the smallest multiple of m not in $\overline{f^\infty(T)}$. Taking $a = m$ and $b = (\ell-1)m$ in Claim 2 finishes.

This completes the proof.

Remark. If all multiples of m are in $\overline{f^\infty(T)}$, then Claim 1 establishes that $f(T) = f^2(T)$. Otherwise, the argument in Claim 1 restricted to $[1, \ell m]$ shows that $f(T) \cap [1, \ell m] = f^\infty(T) \cap [1, \ell m]$, i.e. we may take $j = 1$ in Claim 2. This establishes that $f^4(T) = f^5(T)$ for all T.

Remark. It has been claimed by Colin Tang (independently), Justin Lee, and Espen Slettnes that $f^3(T) = f^4(T)$ for all T, but I have yet to review a proof of this claim.

§2.6 Solutions to Mock IMO

MIMO1 — ISL 2022 A2

Let $k \geq 2$ be an integer. A nonempty set S of real numbers has the property that every element $s \in S$ can be written as the sum of k distinct elements of S that are not equal to s. Find the smallest possible value of $|S|$, in terms of k.

The smallest value of $|S|$ is $k+4$.

Lower bound: Let M be the maximal element of S and let A be the set of size k not containing M that sums to M. Similarly let m be the minimal element and B the set summing to m.

Let $\sigma(T)$ denote the sum of the elements of set T. Consider the sets

$$\begin{aligned} X &= S \setminus (A \sqcup \{M\}) \\ \text{and} \quad Y &= S \setminus (B \sqcup \{m\}). \end{aligned}$$

Then we have

$$\begin{aligned} |X| = |Y| &= |S| - (k+1), \\ \sigma(X) &= \sigma(S) - 2M, \\ \text{and} \quad \sigma(Y) &= \sigma(S) - 2m. \end{aligned}$$

In particular,

$$\sigma(Y) - \sigma(X) = 2(M-m).$$

But X and Y have the same size and each contain elements in the range $[m, M]$, so we require $|X| = |Y| \geq 3$, implying $|S| \geq k+4$.

Construction: For each $\ell \geq 3$, we may check that $S = \{\pm 1, \pm 2, \dots, \pm \ell\}$ works for $k = 2\ell - 4$ and $S = \{0, \pm 1, \dots, \pm \ell\}$ works for $k = 2\ell - 3$.

MIMO2 — ISL 2022 A6

Find all rational numbers q for which there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + f(y)) = f(x) + f(y) \quad \text{and} \quad f(z) \neq qz$$

for all real numbers x, y, z.

The q tht fail are of the form $\frac{n+1}{n}$ for some integer n. The rest work.

Proof of necessity: Setting $x = nf(0)$ and $y = 0$ gives $f((n+1)f(0)) = f(nf(0)) + f(0)$, implying inductively that $f(nf(0)) = (n+1)f(0)$ for all integers n. If $q = \frac{n+1}{n}$, then taking $z = nf(0)$ gives $f(z) = qz$.

Proof of sufficiency: If $q \neq \frac{n+1}{n}$ for any n, it is equivalent to say $\frac{1}{q-1} \notin \mathbb{Z}$. In particular, any arithmetic progression with common difference $q-1$ does not hit every (positive) integer. (If $q = 1$, take $f(x) = x + 1$. Henceforth $q \neq 1$.)

Define f as follows:

(i) For $r \in [0, 1)$, take $f(r)$ as the smallest positive integer not in the progression $\ldots$, a_{-2}, a_{-1}, a_0, a_1, a_2, $\ldots$ defined by $a_n = qr + (q-1)n$.

(ii) For all x, take $f(x) = \lfloor x \rfloor + f(x - \lfloor x \rfloor)$.

Since $f(y)$ is always an integer, it is easy to check f satisfies the functional equation.

Moreover, if $f(z) = qz$ and $z = n + r$ where $n = \lfloor z \rfloor$ and $r \in [0, 1)$, then

$$n + f(r) = q(n + r) \implies f(r) = qr + (q-1)n,$$

which is a situation we explicitly avoid.

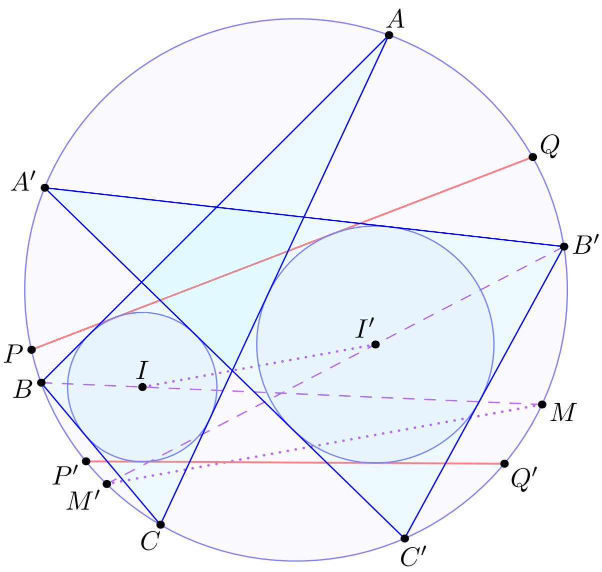
MIMO3 — ISL 2022 G8

Let $AA'BCC'B'$ be a convex cyclic hexagon such that line AC is tangent to the incircle of $\triangle A'B'C'$ and line $A'C'$ is tangent to the incircle of $\triangle ABC$. Let lines AB and $A'B'$ intersect at X and let lines BC and $B'C'$ intersect at Y. Prove that if $XBYB'$ is a convex quadrilateral, then it has an incircle.

Let Ω be the circumcircle, and let ω and ω' denote the incircles of $\triangle ABC$ and $\triangle A'B'C'$.

Claim 1. B, B', I, I' are concyclic.

Proof. Let $\overline{BI}$ and $\overline{B'I'}$ intersect Ω again at M and M'. It is clear $\overline{MM'} \parallel \overline{II'}$ (since both make equal angles with $\overline{AC}$ and $\overline{A'C'}$), so the concyclicity follows from Reim. $\square$



Let a common external tangent to ω and ω' intersect Ω at P and Q.

Claim 2. P, Q, I, I' are concyclic.

Proof. By Poncelet's porism there are R and R' on Ω so that $\triangle PQR$ has incircle ω and $\triangle PQR'$ has incircle ω'. Then

$$\angle PIQ = 90^\circ + \tfrac{1}{2}\angle PRQ = 90^\circ + \tfrac{1}{2}\angle PR'Q = \angle PI'Q,$$

implying the concyclicity. $\square$

Let $\overline{P'Q'}$ be the other common external tangent, and let $X = \overline{II'} \cap \overline{PQ} \cap \overline{P'Q'}$ be the exsimilicenter of ω and ω'. It follows from $PQII'$ and $P'Q'II'$ cyclic that there is an inversion Ψ at X swapping (I, I'), (P, Q), (P', Q'). This inversion must also swap (B, B').

In particular, $\overline{BB'}$ passes through the exsimilicenter X of ω and ω'. By converse Monge (say, via a phantom circle argument), we conclude.

> **Remark.** A solution by Fedir Yudin constructs $J = \overline{BI} \cap \overline{B'I'}$ and shows via a length computation that J is equidistant from the four lines. There is also an approach with Desargue involution.

MIMO4 — ISL 2022 G2

Point P lies in the interior of acute triangle ABC such that lines AP and BC are perpendicular. Points D and E on side BC satisfy $\overline{PD} \parallel \overline{AC}$ and $\overline{PE} \parallel \overline{AB}$, and points $X \neq A$ and $Y \neq A$ lie on the circumcircles of $\triangle ABD$ and $\triangle ACE$, respectively, such that $DA = DX$ and $EA = EY$. Prove that points B, C, X, and Y are concyclic.

Note the foot from A to $\overline{BC}$ has equal power to (ABD) and (ACE), thus the altitude is the radical axis and so $A' = \overline{BX} \cap \overline{CY}$ the reflection of A in $\overline{BC}$ does as well. This finishes.

MIMO5 — ISL 2022 N5

For each $1 \le i \le 9$ and positive integer T, let $d_i(T)$ denote the total number of times the digit i appears when all multiples of 2023 between 1 and T inclusive are written out in base 10. Prove that there are infinitely many positive integers T such that there are exactly two distinct values among $d_1(T)$, $d_2(T)$, …, $d_9(T)$.

Let $T = 10^{n\varphi(2023)} - 1$ for any positive integer n. Let $d_{ij}(T)$ denote the number of times the digit i appears in the jth digit (where $j = 0$ denotes the units digit, $j = 1$ denotes the tens digit, and so on).

> **Claim 1.** There are two distinct values among $d_{10}(T)$, $d_{20}(T)$, …, $d_{90}(T)$.

Proof. The units digits 3, 6, 9 each appear one more than each of the remaining digits. □

> **Claim 2.** For each i, we have $d_{i0}(T) = d_{i1}(T) = d_{i2}(T) = \cdots$.

Proof. For each n with $2023 \mid n$ whose $n\varphi(2023)$th digit from the right is equal to d, note that the cyclic shift $10n - dT$ is also divisible by 2023. Hence if we split the numbers between 1 and T into equivalence classes based on cyclic shifts, each equivalence class contributes to d_{i0}, d_{i1}, … equally. This proves the claim. □

The two above claims combined solve the problem.

MIMO6 — ISL 2022 C7

Let s be a positive integer. Lucy and Lucky play the following game on a blackboard. Lucy initially writes s integer-valued 2023-tuples on the board. Lucky then gives Lucy an integer-valued 2023-tuple. Afterwards, Lucy can repeatedly take any two (not necessarily distinct) tuples $(v_1, \dots, v_{2023})$ and $(w_1, \dots, w_{2023})$ on the blackboard and writes the tuples

$$(v_1 + w_1, \dots, v_{2023} + w_{2023}) \quad \text{and} \quad (\max(v_1, w_1), \dots, \max(v_{2023}, w_{2023}))$$

on the board. Lucy wins if she can write Lucky's tuple on the board in a finite number of steps.
Determine the smallest value of s for which Lucy has a winning strategy.

TODO