Blue Group Tests

MOP 2021

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Contents

§1 Problems

§1.1 MOP Quiz 1

Problem B1.1 (Bulgaria TST 2004/3/2)**.** Let $n \geq 4$ be a positive integer. The $\binom{2n}{2}$ edges of a complete graph with $2n$ vertices are to be colored blue and red, in such a way that

- no triangle has all three edges blue; and
- no complete subgraph on n vertices has all its edges red.

In such a coloring, determine the minimum possible number of blue edges.

Problem B1.2 (Brazil 2012)**.** Find the least nonnegative integer b such that there exists a nonnegative integer n for which the last 2021 decimal digits of b^n are all 1.

§1.2 MOP Test 2

Problem B2.1. Does there exist a positive integer m for which the equation

$$(a^3 - a)(b^3 - b) = mc^2$$

has infinitely many positive integer solutions (a, b, c) in which $a \neq b$?

Problem B2.2 (ISL 2020 A5)**.** A magician wishes to perform the following trick.

> The magician announces a positive integer n, along with $2n$ real numbers $x_1 < \cdots < x_{2n}$ to the audience. Then an audience member secretly chooses a polynomial $P(x)$ of degree n with real coefficients and gives the magician the values $P(x_1)$, ..., $P(x_{2n})$ in any order. After that, the magician announces the polynomial $P(x)$.

Can the magician perform this trick?

Problem B2.3 (ISL 2020 G7)**.** Let P be a point on the circumcircle of acute triangle ABC. Let D, E, and F be the reflections of P in the A-midline, B-midline, and C-midline. Let ω be the circumcircle of the triangle formed by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

Show that the circumcircles of $\triangle ADP$, $\triangle BEP$, $\triangle CFP$, and ω share a common point.

§1.3 MOP Test 3

Problem B3.1 (ISL 2020 N4)**.** For any odd prime p and integer n, let $d_p(n)$ denote the remainder when n is divided by p. The sequence $(x_0, x_1, \ldots)$ is a *p-dop* if x_0 is a positive integer coprime to p, and $x_{n+1} = x_n + d_p(x_n)$ for all $n \geq 0$. Do there exist infinitely many primes p such that there exist p-dops $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ for which...

(a) ... $a_n < b_n$ infinitely often and $a_n > b_n$ infinitely often?

(b) ... $a_0 < b_0$, but $a_n > b_n$ for all $n \geq 1$.

Problem B3.2 (ISL 2020 G6)**.** Let ABC be a triangle with $AB < AC$, incenter I, and A-excenter I_A. The incircle meets $\overline{BC}$ at D. Define $E = \overline{AD} \cap \overline{BI_A}$ and $F = \overline{AD} \cap \overline{CI_A}$. Show that the circumcircles of $\triangle AID$ and $\triangle I_AEF$ are tangent to each other.

Problem B3.3 (ISL 2020 A2')**.** Let $\mathcal{A}$ denote the set of polynomials in 100 variables x_1, ..., x_{100} with integer coefficients.

(a) Prove that any monomial $x_1^{e_1} x_2^{e_2} \cdots x_{100}^{e_{100}}$ with $e_1 + e_2 + \cdots + e_{100} \geq 4951$ can be expressed in the form

$$p_1q_1 + p_2q_2 + \cdots + p_{100}q_{100}$$

where $p_i, q_i \in \mathcal{A}$ for all i, and q_i is a symmetric polynomial satisfying $q_i(0, \ldots, 0)$ for all i.

(b) Prove that $x_2^1 x_3^2 \cdots x_{100}^{99}$ cannot be expressed in this way.

§1.4 MOP Test 4

Problem B4.1. Can you find 15 positive integers (not necessarily distinct) with product k, such that if each of the integers is increased by 1, the new product is $2021k$?

Problem B4.2 (ISL 2020 C4)**.** The Fibonacci numbers F_0, F_1, ... are defined by $F_0 = 0$, $F_1 = 1$, and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$.

Let $n \geq 2$ be a fixed integer and suppose that S is a set of integers such that each element of $\{F_2, F_3, \ldots, F_n\}$ can be written as the difference of two elements in S. How small can $|S|$ be?

Problem B4.3 (ISL 2020 N6)**.** For a positive integer n,

- let $d(n)$ be the number of positive integer divisors of N, and
- let $\varphi(n)$ be the number of positive integers at most n which are relatively prime to n.

Does there exist a constant C such that

$$\frac{\varphi(d(n))}{d(\varphi(n))} \leq C$$

for all $n \geq 1$?

§1.5 MOP Quiz 5

Problem B5.1 (Brazil Undergrad 2010 Olympiad)**.** Let k be a positive integer for which $p = 60k + 7$ is prime. Suppose that p divides $10^{2n} + 8 \cdot 10^n + 1$ for some positive integer n. Show that k and n are even.

Problem B5.2 (ISL 2020 G5)**.** Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ such that $KLMN$ is a rhombus with $\overline{KL} \parallel \overline{AC}$ and $\overline{LM} \parallel \overline{BD}$. Let ω_A, ω_B, ω_C, ω_D be the incircles of $\triangle ANK$, $\triangle BKL$, $\triangle CLM$, $\triangle DMN$. Prove that the common internal tangents to ω_A and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

§1.6 ELMO

Problem ELMO1 (Eric Shen)**.** Let ABC be a triangle, and let P and Q lie on sides AB and AC such that the circumcircle of $\triangle APQ$ is tangent to segment BC at a point D. Let E lie on segment BC such that $BD = EC$. Line DP intersects the circumcircle of $\triangle CDQ$ again at X, and line DQ intersects the circumcircle of $\triangle BDP$ again at Y. Prove that points D, E, X, Y are concyclic.

Problem ELMO2 (Maxim Li)**.** Let $n \geq 2$ be an integer and let a_1, a_2, …, a_n be integers such that $n \mid a_i - i$ for all integers $1 \leq i \leq n$. Prove there exists an infinite sequence b_1, b_2, … with $b_i \in \{a_1, a_2, \ldots, a_n\}$ for each i, such that

$$\sum_{i=1}^{\infty} \frac{b_i}{n^i} \in \mathbb{Z}.$$

Problem ELMO3 (Maxim Li)**.** Each cell of a 100×100 grid is colored with one of 101 colors. A cell is *diverse* if, among the 199 cells in its row and column, every color appears at least once. Determine the maxmum possible number of diverse cells.

Problem ELMO4 (Brandon Wang)**.** Suppose the set of positive integers is partitioned into $n \geq 2$ disjoint arithmetic progressions S_1, S_2, …, S_n with common differences d_1, d_2, …, d_n. Prove that there exists exactly one index $1 \leq i \leq n$ such that

$$\prod_{j \neq i} d_j \in S_i.$$

Problem ELMO5 (Sean Li)**.** Let n and k be positive integers. Two infinite sequences (s_i) and (t_i) are *equivalent* if $s_i = s_j$ if and only if $t_i = t_j$ for all positive integers i and j, and a sequence (t_i) has *equi-period* k if t_1, t_2, … and t_{k+1}, t_{k+2}, … are equivalent. In terms of n and k, how many sequences of equi-period k are there in the set of sequences with each entry in the set $\{1, 2, \ldots, n\}$, up to equivalence?

Problem ELMO6 (Maxim Li)**.** In triangle ABC, points D, E, F lie on segments BC, CA, AB, respectively, such that each of the quadrilaterals $AFDE$, $BDEF$, $CEFD$ has an incircle. Prove that the inradius of $\triangle ABC$ is twice the inradius of $\triangle DEF$.

§1.7 Mock IMO

Problem MIMO1 (ISL 2020 C2)**.** In a regular 100-gon, 41 vertices are colored black and the other 59 vertices are colored white. A quadrilateral is *weird* if it has three vertices of one color and one vertex of the other color.

Prove that there exist 24 pairwise disjoint weird quadrilaterals. (Two quadrilaterals are *disjoint* if they have no common vertices and their interiors do not intersect.)

Problem MIMO2 (ISL 2020 A3)**.** Let a, b, c, d be positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Problem MIMO3 (ISL 2020 N7)**.** Let S be a set of $n \geq 3$ positive integers, none of which is the sum of two different numbers in S. Prove that there exists a permutation of S in which none of the middle $n-2$ integers divides the sum of its neighbors.

Problem MIMO4 (ISL 2020 G3)**.** Let $ABCD$ be a convex quadrilateral with $\min\{\angle B, \angle D\} > 90^\circ$ and $\angle A = \angle C$. Points E and F are the reflections of A in $\overline{BC}$ and $\overline{CD}$. Segments AE and AF meet line BD at K and L.

Prove that the circumcircles of $\triangle BEK$ and $\triangle DFL$ are tangent to each other.

Problem MIMO5 (ISL 2020 N5)**.** Determine all functions $f : \{1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ such that

- $f(xy) = f(x) + f(y)$ for all positive integers x and y, and
- there exists an infinite set S of positive integers such that $f(a) = f(b)$ whenever $a + b \in S$.

Problem MIMO6 (ISL 2020 C8)**.** Anastasia and Bananastasia play a game on a board as follows. Initially, the board contains 2020 copies of the number 1. Each round proceeds as follows:

1. Anastasia erases two numbers x and y from the board.
2. Bananastasia writes one of $x + y$ and $|x - y|$ on the board.

After each round, the game ends if one of the following holds:

- one number on the board is larger than the sum of all other numbers on the board, or
- all numbers on the board are zeroes.

After the game ends, Bananastasia must give Anastasia one slice of banana bread for every number remaining on the board. How many slices of banana bread can Anastasia guarantee, assuming optimal play from both players?

§1.8 MOP Quiz 6

Problem B6.1 (ISL 2020 G1)**.** Let ABC be an isosceles triangle with $CA = CB$, and let D be a point on side AB with $AD < DB$. Let P and Q be the projections from D to $\overline{CB}$ and $\overline{CA}$. The perpendicular bisector of $\overline{PQ}$ meets segment CQ at E, and the circumcircles of $\triangle ABC$ and $\triangle CPQ$ meet at $F \neq C$.

Show that if P, E, F are collinear, then $\angle C = 90°$.

Problem B6.2 (ISL 2020 A1)**.** Let n be a positive integer. Determine the smallest real number C such that, for all real x,

$$\sqrt[n]{\frac{x^{2n}+1}{2}} \leq C(x-1)^2 + x.$$

§1.9 MOP Test 7

Problem B7.1 (ISL 2020 G4)**.** Let $n \geq 6$ be an integer and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be pairwise disjoint closed disks in the plane with radii $R_1 \geq \cdots \geq R_n$. For each $i \in \{1, \ldots, n\}$, let P_i be a point on $\mathcal{D}_i$. Let O be a point in the plane. Prove that

$$OP_1 + OP_2 + \cdots + OP_n \geq R_6 + R_7 + \cdots + R_n.$$

Problem B7.2 (ISL 2020 C5)**.** Let p be an odd prime, let $N = \frac{1}{4}(p^3 - p) - 1$, and let S be a subset of $\{1, \ldots, N\}$. Show that there exists an integer $a \in \{1, \ldots, p-1\}$ such that for all positive integers $n \in N$,

$$\frac{|S \cap \{1, \ldots, n\}|}{n} \neq \frac{a}{p}.$$

Problem B7.3 (ISL 2020 G8)**.** Let ABC be a triangle with incenter I and circumcircle Γ. Circles ω_B passing through B and ω_C passing through C are tangent at I. Let ω_B meet minor arc AB of Γ at P and $\overline{AB}$ at $M \neq B$, and let ω_C meet minor arc AC of Γ at Q and $\overline{AC}$ at $N \neq C$, Rays PM and QN meet at X. Let Y be a point such that $\overline{YB}$ is tangent to ω_B and $\overline{YC}$ is tangent to ω_C.

Show that A, X, Y are collinear.

§2 Solutions

§2.1 Solutions to MOP Quiz 1

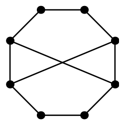
Problem B1.1 (Bulgaria TST 2004/3/2)
Let $n \geq 4$ be a positive integer. The $\binom{2n}{2}$ edges of a complete graph with $2n$ vertices are to be colored blue and red, in such a way that

- no triangle has all three edges blue; and
- no complete subgraph on n vertices has all its edges red.

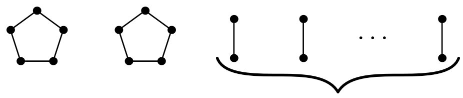
In such a coloring, determine the minimum possible number of blue edges.

The answer is 10 for $n = 4$ and $n + 5$ for $n \geq 5$. In what follows, we only draw blue edges and omit the red edges. The condition is that there are no triangles, and no independent set of size n.

Construction: For $n = 4$, consider



For $n \geq 5$, consider



$n-5$

Bound: One may show via exhaustion that:

- In a triangle-free connected graph on 6 vertices there is an independent set of size 3.
- In a triangle-free connected graph on 8 vertices and ≤ 9 edges, there is an independent set of size 4. (This is very annoying.)
- In a triangle-free connected graph on $k \in \{3, 5, 7\}$ vertices, we can find an independent set of size $\lfloor k/2 \rfloor$.

These already solve $n = 4$, so we focus on $n \geq 5$.

Say a graph with k vertices is *spacious* if there is an independent set of size $\geq k/2$. Let the graph contain A spacious connected components and B non-spacious connected components. Do not count singletons. (Observe that non-spacious components must contain an odd cycle, so they have size ≥ 5.)

One can easily establish the bounds $2A + 5B \leq 2n$ and #edges $\geq 2n - A$, so if $A \leq n - 5$ we are already done. Henceforth assume $A \geq n - 4$, implying $B = 1$. (Of course, if $B = 0$ we are trivially done by choosing independent subsets from each connected component.)

Consider this single non-spacious component. Since $A \geq n - 4$, this component contains at most 8 vertices.

- If it contains an odd number of vertices, there must also be a singleton in the graph by parity. Then choosing this singleton along with almost half the vertices in this component (by the observations we made earlier) allows us to find a size-n independent set of the original graph, contradiction.

- If it contains an even number of vertices, by the observations it has 8 vertices and ≥ 10 edges. It readily follows that #edges $\geq 2n - A + 2 \geq n + 6$, so we are done.

Problem B1.2 (Brazil 2012)

Find the least nonnegative integer b such that there exists a nonnegative integer n for which the last 2021 decimal digits of b^n are all 1.

Mod spamming shows that the problem condition is equivalent to

- $b \equiv 7 \pmod{16}$,
- $b \equiv 1 \pmod{5}$, and
- $b \not\equiv 1 \bmod 25$.

§2.2 Solutions to MOP Test 2

Problem B2.1

Does there exist a positive integer m for which the equation

$$(a^3 - a)(b^3 - b) = mc^2$$

has infinitely many positive integer solutions (a, b, c) in which $a \neq b$?

Yes, $m = 2$ works.

First, we may generate infinitely many solutions (u, v) to $u^2 - 8v^2 = 9$, with u odd, by starting with $(9, 3)$ and using $(u, v) \mapsto (3u + 8v, u + 3v)$.

For each such pair, $(a, b, c) = (\frac{u-1}{2}, \frac{u+1}{2}, \cdot \frac{u^2-1}{4} \cdot v)$ works by noting

$$\begin{aligned}(a^3 - a)(b^3 - b) &= \left(\tfrac{u-3}{2} \cdot \tfrac{u-1}{2} \cdot \tfrac{u+1}{2}\right) \cdot \left(\tfrac{u-1}{2} \cdot \tfrac{u+1}{2} \cdot \tfrac{u+3}{2}\right) \\ &= \left(\tfrac{u^2-1}{4}\right)^2 \cdot \tfrac{u^2-9}{4} \\ &= \left(\tfrac{u^2-1}{4}\right)^2 \cdot 2v^2.\end{aligned}$$

Problem B2.2 (ISL 2020 A5)

A magician wishes to perform the following trick.

> The magician announces a positive integer n, along with $2n$ real numbers $x_1 < \cdots < x_{2n}$ to the audience. Then an audience member secretly chooses a polynomial $P(x)$ of degree n with real coefficients and gives the magician the values $P(x_1)$, ..., $P(x_{2n})$ in any order. After that, the magician announces the polynomial $P(x)$.

Can the magician perform this trick?

No, the magician cannot perform this trick.

The audience member has enough freedom to select a polynomial P of degree $\leq n$ with the property that

$$P(x_1) + P(x_2) = P(x_3) + P(x_4) = \cdots = P(x_{2n-1}) + P(x_{2n}) = 0.$$

(Indeed, the coefficients will generate a homogeneous system of n linear equations with $n+1$ variables, which has a nontrivial solution.)

Now there are roots in the intervals $[x_1, x_2]$, ..., $[x_{2n-1}, x_{2n}]$, so $\deg P = n$. Finally the values $P(x_1)$, ..., $P(x_{2n})$ are not all zero, so the magician cannot discern between P and $-P$.

Problem B2.3 (ISL 2020 G7)

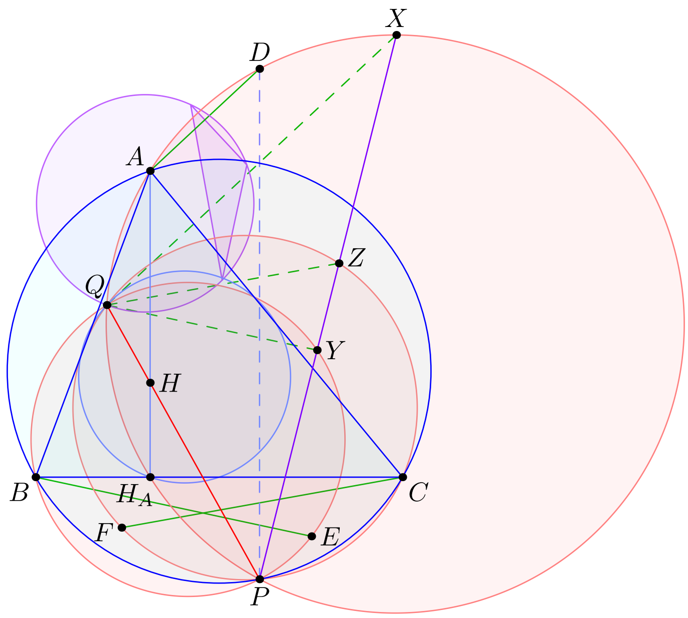
Let P be a point on the circumcircle of acute triangle ABC. Let D, E, and F be the reflections of P in the A-midline, B-midline, and C-midline. Let ω be the circumcircle of the triangle formed by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

Show that the circumcircles of $\triangle ADP$, $\triangle BEP$, $\triangle CFP$, and ω share a common point.

Let H be the orthocenter and $H_AH_BH_C$ the orthic triangle. Let the negative inversion at H sending $\triangle ABC$ to $\triangle H_AH_BH_C$ send P to Q. I claim Q is the desired concurrence point.

First, Q lies on the circles (ADP), (BEP), (CFP) by noting

$$HP \cdot HQ = HA \cdot HH_A = HB \cdot HH_B = HC \cdot HH_C.$$



To show Q lies on ω, it will suffice to show the Steiner line exists, i.e. the reflections X, Y, Z of Q over the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are collinear. In fact, I claim P, X, Y, Z are all collinear.

Observe that $\measuredangle XPD = \measuredangle APQ = \measuredangle APH$, so

$$\begin{aligned}\measuredangle YPZ &= \measuredangle YPE + \measuredangle FPZ + \measuredangle EPF \\ &= \measuredangle BPH + \measuredangle HPC + \measuredangle CAB = 0^\circ.\end{aligned}$$

By symmetry we are done.

§2.3 Solutions to MOP Test 3

Problem B3.1 (ISL 2020 N4)

For any odd prime p and integer n, let $d_p(n)$ denote the remainder when n is divided by p. The sequence $(x_0, x_1, \ldots)$ is a *p-dop* if x_0 is a positive integer coprime to p, and $x_{n+1} = x_n + d_p(x_n)$ for all $n \geq 0$. Do there exist infinitely many primes p such that there exist p-dops $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ for which...

(a) ... $a_n < b_n$ infinitely often and $a_n > b_n$ infinitely often?

(b) ... $a_0 < b_0$, but $a_n > b_n$ for all $n \geq 1$.

The answer to both parts is yes.

Solution (a) Any prime $p \equiv 3, 5 \pmod 8$ works. For such p, two is not a quadratic residue, so $2^{\frac{p-1}{2}} \equiv -1 \pmod p$.

I claim the p-dops starting at 2 and $p-2$ work. Observe $a_i \equiv 2^{i+1} \pmod p$ and $b_i \equiv -2^{i+1} \pmod p$ for each i. Then $a_{i+(p-1)/2} = b_i$ for each i, so

$$d(a_0) + \cdots + d(a_{p-2}) = d(b_0) + \cdots + d(b_{p-2}) \equiv 0 \pmod p.$$

Observe, then, that for some N, we have $a_{p-2} = Np + 1$, $a_{p-1} = Np + 2$, $b_{p-2} = Np - 1$, $b_{p-1} = Np + p - 2$.

Moreover the sequences $(a_{p-1}, a_p, \ldots)$ and $(b_{p-1}, b_p, \ldots)$ are the sequences $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ shifted by Np, so we always have $a_{k(p-1)+p-2} > b_{k(p-1)+p-2}$ and $a_{k(p-1)+p-1} < b_{k(p-1)+p-1}$, as desired.

Solution (b) By Kobayashi infinitely many primes divide the sequence

$$2^1 - 1, \quad 2^3 - 1, \quad 2^5 - 1, \quad \ldots.$$

For such primes p, we have $2^{\text{odd}} \equiv 1 \pmod p$, so $m := \operatorname{ord}(2 \bmod p)$ is odd.

If we consider equivalence classes modulo p where r and s are in the same class if r/s is a power of two. Each class contains m elements, so there are $(p-1)/m$ classes. The sum of the elements in all classes is $p(p-1)/2$, and since m is odd this not a multiple of $(p-1)/m$.

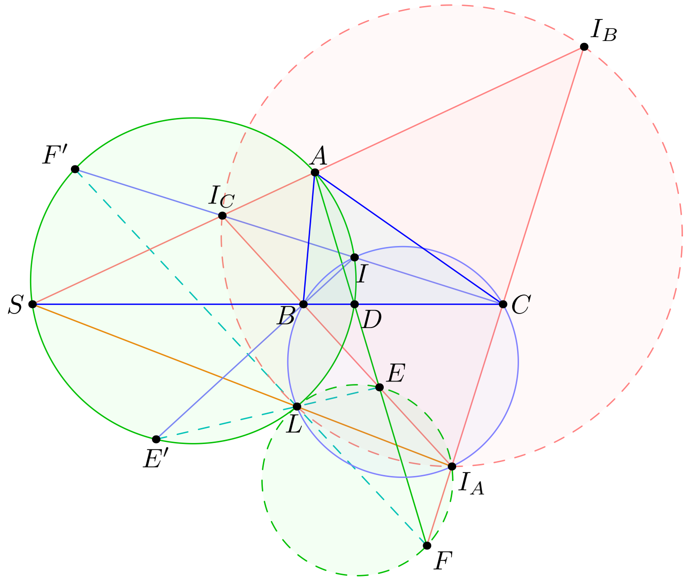
It follows that two equivalence classes have different sums, so for some two starting indices a_0 and b_0, we have $d(a_0) + \cdots + d(a_{p-2})$ and $d(b_0) + \cdots + d(b_{p-2})$ are different multiples of p. Without loss of generality the former is larger than the latter.

Then start at a_0 and $N + b_0$ for some very large N, so initially the first sequence is larger, but eventually the second is larger. Then there is some maximal index i with $a_i < b_i$, so starting instead at a_i and b_i solves the problem.

Problem B3.2 (ISL 2020 G6)

Let ABC be a triangle with $AB < AC$, incenter I, and A-excenter I_A. The incircle meets $\overline{BC}$ at D. Define $E = \overline{AD} \cap \overline{BI_A}$ and $F = \overline{AD} \cap \overline{CI_A}$. Show that the circumcircles of $\triangle AID$ and $\triangle I_AEF$ are tangent to each other.

Let I_B and I_C be the $B-$ and C-excenters, and let $S = \overline{BC} \cap \overline{I_BI_C}$. Let $(I_AI_BI_C)$ and $(BICI_A)$ intersect again at L. By radical axis theorem I_A, L, S collinear, and also $\angle IAS = \angle IDS = \angle ILS = 90^\circ$, so A, I, D, L, S are concyclic.



Let lines BI and CI meet (AID) again at E' and F'. Since $AIBI_C$ is cyclic, we have $\overline{SE'} \parallel \overline{BI_C}$ by Reim's. Analogously $\overline{SF'} \parallel \overline{CI_B}$.

Finally,

$$\begin{aligned}\measuredangle I_ALE &= \measuredangle I_AFE = \measuredangle CI_BI_C + \measuredangle SAD \\ &= \measuredangle DIB + \measuredangle SID = \measuredangle SIE' = \measuredangle SLE',\end{aligned}$$

so $L \in \overline{EE'}$, and similarly $L \in \overline{FF'}$. A homothety at L sends (AID) to (I_AEF), so they are tangent at L.

Problem K3.3 (ISL 2020 A2')

Let $\mathcal{A}$ denote the set of polynomials in 100 variables $x_1, \dots, x_{100}$ with integer coefficients. Find the smallest integer N such that any monomial $x_1^{e_1}x_2^{e_2}\cdots x_{100}^{e_{100}}$ with $e_1+e_2+\cdots+e_{100} \geq N$ can be expressed in the form

$$p_1q_1 + p_2q_2 + \cdots + p_{100}q_{100},$$

where $p_i, q_i \in \mathcal{A}$ for all i, and q_i is a symmetric polynomial satisfying $q_i(0, 0, \dots, 0) = 0$ for all i.

The answer is $\binom{100}{2} + 1 = 4951$.

Proof $N \geq 4951$ work: I contend we can express all monomials of degree $e_1+\cdots+e_{100} \geq 4951$ in the form

$$p_1S_1 + p_2S_2 + \cdots + p_{100}S_{100},$$

where S_i is the ith elementary symmetric polynomial ($S_1 = x_1 + \cdots + x_{100}$, $S_2 = x_1x_2 + \cdots + x_{99}x_{100}$, etc.).

Say the *character* of a monomial $x_1^{e_1}\cdots x_{100}^{e_{100}}$ is $e_1^2 + e_2^2 + \cdots + e_{100}^2$. We will strong induct on character, with the base case as follows: If all 100 variables appear with positive exponent, we are done by factoring out $x_1x_2\cdots x_{100}$.

Now assume without loss of generality $e_1 \geq e_2 \geq \cdots \geq e_{100} = 0$. We can find an i with $e_{i+1} \leq e_i - 2$; otherwise, $e_{100} = 0$, $e_{99} \leq 1$, $e_{98} \leq 2$, $\dots$, and thus $e_1 + \cdots + e_{100} \leq 4950$, contradiction. Then subtract off

$$x_1^{e_1-1}x_2^{e_2-1}\cdots x_i^{e_i-1}x_{i+1}^{e_{i+1}}x_{i+2}^{e_{i+2}}\cdots x_{100}^{e_{100}} \cdot S_i.$$

The remaining monomials have strictly smaller character, so by the inductive hypothesis they are already expressible in the desired form.

Proof $N = 4950$ fails: I claim the monomial $x_2^1 x_3^2 \cdots x_{100}^{99}$ does not have this property.

Decompose the polynomials p_i into monomials and q_i into homogeneous polynomials, so we have

$$x_2^1 x_3^2 \cdots x_{100}^{99} = \sum_i p_i(x_1, \ldots, x_{100}) \cdot q_i(x_1, \ldots, x_{100}),$$

with p_i monomials and q_i homogeneous and symmetric. Ignore all terms where $\deg p_i + \deg q_i \neq 4950$, so all terms in the expansion have degree 4950.

Now consider the following, where we sum over permutations π of $\{1, \ldots, 100\}$:

$$\sum_\pi \operatorname{sgn}(\pi) \cdot x_{\pi(2)}^1 \cdots x_{\pi(100)}^{99} = \sum_i q_i(x_1, \ldots, x_{100}) \cdot \sum_\pi p_i(x_{\pi(1)}, \ldots, x_{\pi(100)}) \cdot \operatorname{sgn}(\pi),$$

However, observe the following:

Claim. If p_i is a monomial with $\deg p_i \leq 4949$, then

$$\sum_\pi p_i(x_{\pi(1)}, \ldots, x_{\pi(100)}) \cdot \operatorname{sgn}(\pi) = 0.$$

Proof. Two exponents must be equal; otherwise, $\deg p_i \geq 0 + 1 + \cdots + 99 = 4950$, contradiction. If the exponents of x_i and x_j are equal, we can pair permutations that swap i and j, which will cancel out in the summation. □

It follows that

$$\sum_\pi \operatorname{sgn}(\pi) \cdot x_{\pi(2)}^1 \cdots x_{\pi(100)}^{99} \equiv 0,$$

which is absurd.

§2.4 Solutions to MOP Test 4

Problem B4.1

Can you find 15 positive integers (not necessarily distinct) with product k, such that if each of the integers is increased by 1, the new product is $2021k$?

Yes:

$$\left(\frac{2}{1}\right)^{10} \cdot \left(\frac{5}{4}\right)^{3} \cdot \frac{101}{100} \cdot \frac{2021}{2020} = 2021.$$

Problem B4.2 (ISL 2020 C4)

The Fibonacci numbers F_0, F_1, … are defined by $F_0 = 0$, $F_1 = 1$, and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$.

Let $n \geq 2$ be a fixed integer and suppose that S is a set of integers such that each element of $\{F_2, F_3, \ldots, F_n\}$ can be written as the difference of two elements in S. How small can $|S|$ be?

The answer is $k = \lceil \frac{n}{2} \rceil + 1$, achieved by $S = \{F_0, F_2, F_4, \ldots, F_{2k-2}\}$.

First solution Draw a graph G on the elements of S where we connect u and v whenever $|u-v| \in \{F_1, F_3, F_5, \dots\}$ is a Fibonacci number with odd index. Thus we need at least $\lceil \frac{n}{2} \rceil$ edges drawn.

Now I claim this graph is acyclic: consider a cycle with longest edge F_t, and let the other edge lengths be $F_{x_1}, \dots, F_{x_k}$. Observe that we can write

$$F_t = \pm F_{x_1} \pm F_{x_2} \pm \cdots \pm F_{x_k} \le F_1 + \cdots + F_{t-2} < F_t,$$

contradiction.

Thus there are at least $\lceil \frac{n}{2} \rceil + 1$ vertices.

Second solution (mine) I will show for each $k \ge 2$, if $|S| = k$, then $S - S$ contains at most $2k-3$ distinct Fibonacci numbers. (If $k = 1$, then at most 0.) To this end, we strong induct on k, with base cases $k = 1$ and $k = 2$ trivial.

Draw a graph G with k vertices representing the k elements of S. For each (distinct) Fibonacci number f, draw an edge between one pair of nodes (u, v) with $|u - v| = f$. (If multiple (u, v) exist, choose one.) It is equivalent to show at most $2k - 3$ edges are drawn for $k \ge 2$ and none for $k = 1$.

Let F_t be the longest edge drawn. I contend that upon deleting the edge F_t and the edge F_{t-1} (if it exists), the graph is disconnected. Indeed, if a path exists between the two original endpoints of the F_t edge, then the sum of the lengths of the edges in this path is at most

$$\sum \text{edge length} \le F_2 + F_3 + \cdots + F_{t-2} < F_t,$$

contradiction.

Now split G into $G_1 \sqcup G_2$ with k_1 and k_2 vertices. Since $k \ge 3$ we assume without loss of generality $k_1 \ge 2$. Then G_1 has at most $2k_1 - 3$ edges by inductive hypothesis, and G_2 has at most $2k_2 - 2$ edges (taking the case $k_2 = 1$ into consideration), so the number of edges in G is

$$\#\text{edges in } G \le 2 + (2k_1 - 3) + (2k_2 - 2) = 2k - 3.$$

Problem B4.3 (ISL 2020 N6)

For a positive integer n,

- let $d(n)$ be the number of positive integer divisors of N, and
- let $\varphi(n)$ be the number of positive integers at most n which are relatively prime to n.

Does there exist a constant C such that

$$\frac{\varphi(d(n))}{d(\varphi(n))} \le C$$

for all $n \ge 1$?

The answer is no.

Let $p_1, \dots, p_m$ be the first m primes, and let $q_1, \dots, q_r$ be primes with the property that $p_m < q_i < 2p_m$. By prime number theorem, we can let r grow sufficiently large.

Consider n of the form

$$n = p_1^{2^{s_1}-1} \cdots p_m^{2^{s_m}-1} q_1 \cdots q_r,$$

Then we have

$$d(n) = 2^{s_1+\cdots+s_m+r} \quad \text{and} \quad \varphi(d(n)) = 2^{s_1+\cdots+s_m+r-1}.$$

Moreover

$$\varphi(n) = p_1^{2^{s_1}-1} \cdots p_m^{2^{s_m}-1}(q_1-1)\cdots(q_r-1).$$

Since $\frac{q_i-1}{2} \le p_m$, the prime factors of $q_i - 1$ are in among $\{p_1, \dots, p_m\}$, so we can write

$$\varphi(n) = p_1^{2^{s_1}-2+f_1} \cdots p_m^{2^{s_m}-2+f_m},$$

whence

$$d(\varphi(n)) = \prod \left(2^{s_i} - 1 + f_i\right).$$

Therefore,

$$\frac{\varphi(d(n))}{d(\varphi(n))} \ge 2^{r-1} \prod \left(\frac{2^{s_i}}{2^{s_i} - 1 + f_i}\right).$$

As the s_i grow large, this approaches 2^{r-1}, which can be arbitrarily large.

> **Remark.** Instead of using prime number theorem, we can prove there are unbounded primes in $[k, 2k)$ as follows: if there are always at most R such primes, then then the sum of the reciprocals of the primes in $[2,4) \cup [4,8) \cup [8,16) \cup \cdots$ is
>
> $$\le \frac{R}{2} + \frac{R}{4} + \frac{R}{8} + \cdots \le R,$$
>
> but it also diverges.

§2.5 Solutions to MOP Quiz 5

> **Problem B5.1** (Brazil Undergrad 2010 Olympiad)
> Let k be a positive integer for which $p = 60k + 7$ is prime. Suppose that p divides $10^{2n} + 8 \cdot 10^n + 1$ for some positive integer n. Show that k and n are even.

If n is odd, then

$$\begin{aligned} 0 &\equiv 10^{2n} + 8 \cdot 10^n + 1 \\ &\equiv (10^n - 1)^2 + 10^{n+1} \pmod p \end{aligned}$$

is the sum of two squares, but $p \nmid 10^{n+1}$ and $p \equiv 3 \pmod 4$, contradiction.

If n is even, then

$$\begin{aligned} 0 &\equiv 10^{2n} + 8 \cdot 10^n + 1 \\ &\equiv (10^n + 1)^2 + 6 \cdot 10^{n+1} \pmod p, \end{aligned}$$

so since $p \nmid 10^{n+1}$, -6 is a quadratic residue.

We can check -1 and 3 are not quadratic residues, so 2 is a quadratic residue, so $p \equiv \pm 1 \pmod 8$. The desired conclusion follows.

> **Problem B5.2** (ISL 2020 G5)
> Let $ABCD$ be a cyclic quadrilateral. Points K, L, M, N are chosen on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ such that $KLMN$ is a rhombus with $\overline{KL} \parallel \overline{AC}$ and $\overline{LM} \parallel \overline{BD}$. Let ω_A, ω_B, ω_C, ω_D be the incircles of $\triangle ANK$, $\triangle BKL$, $\triangle CLM$, $\triangle DMN$. Prove that the common internal tangents to ω_A and ω_C and the common internal tangents to ω_B and ω_D are concurrent.

Let $F = \overline{AB} \cap \overline{CD}$ and $G = \overline{AD} \cap \overline{BC}$.

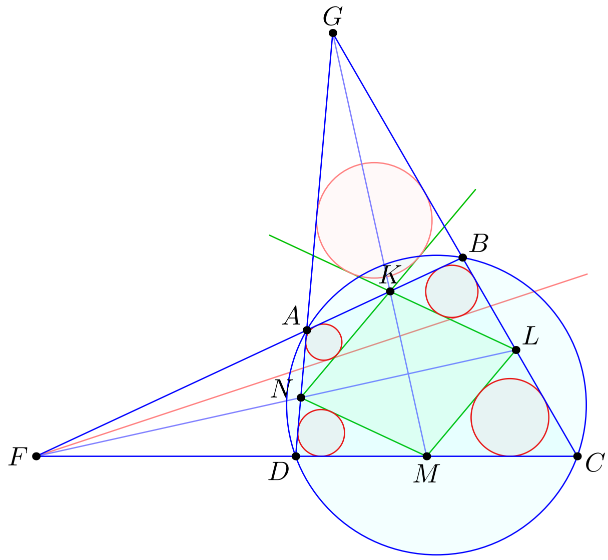
Claim 1. K and M lie on the bisector of $\angle G$.

Proof. Note that $\frac{AK}{AB} = \frac{KN}{BD}$ and $\frac{BK}{AB} = \frac{KL}{AC}$. Dividing, we have

$$\frac{AK}{BK} = \frac{AC}{BD} = \frac{GA}{GB},$$

and similarly for M. $\square$

Analogously F, N, L collinear.



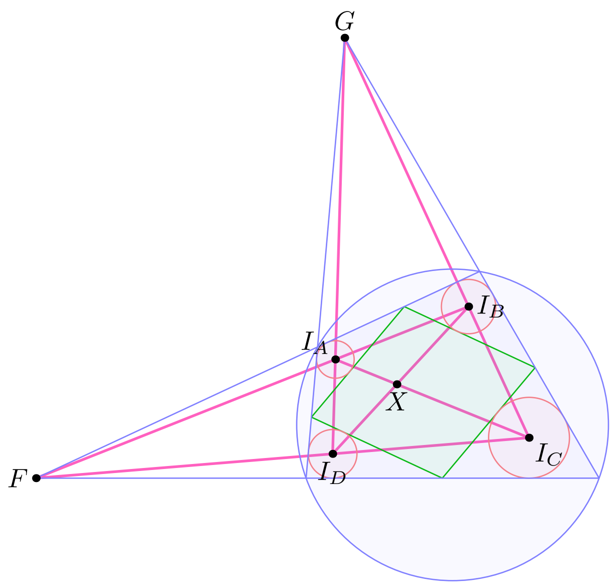
Claim 2. F is the exsimilicenter of ω_A and ω_B.

Proof. Since $\overline{GKM}$ is the perpendicular bisector of $\overline{NL}$, by symmetry there is a circle ω_G tangent to $\overline{GN}$, $\overline{GL}$, $\overline{KN}$, $\overline{KL}$.

By Monge's theorem on ω_A, ω_B, ω_G, the exsimilicenter of ω_A, ω_B lies on line LN, which is sufficient. $\square$

Analogously ω_C, ω_D have exsimilicenter F, and ω_A, ω_D and ω_B, ω_C have exsimilicenter G.

This is enough to solve the problem. Let I_A, I_B, I_C, I_D and r_A, r_B, r_C, r_D be the centers and radii of ω_A, ω_B, ω_C, ω_D, so that $\frac{FI_A}{FI_B} = \frac{r_A}{r_B}$, etc. Also let $X = \overline{I_AI_C} \cap \overline{I_BI_D}$.



To finish, observe:

- By Monge on $(\omega_A, \omega_D, \omega_C)$ and $(\omega_A, \omega_D, \omega_B)$, the insimilicenters of (ω_A, ω_C) and (ω_B, ω_D) lie on the line through F and the insimilicenter of (ω_A, ω_D).
- By Monge on $(\omega_A, \omega_B, \omega_C)$ and $(\omega_A, \omega_B, \omega_D)$, the insimilicenters of (ω_A, ω_C) and (ω_B, ω_D) lie on the line through G and the insimilicenter of (ω_A, ω_B).

Hence the two insimilicenters coincide.

Remark (Generalization)**.** The problem still holds if $KLMN$ is any parallelogram.

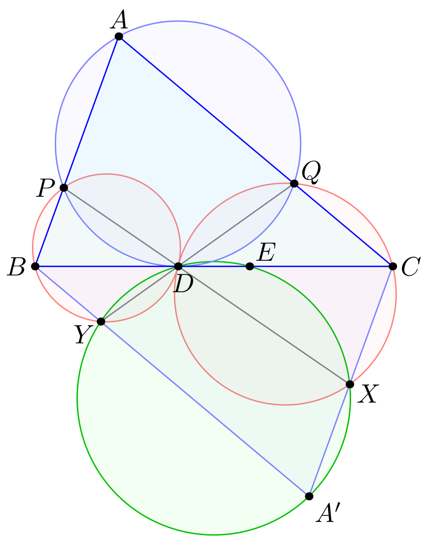
§2.6 Solutions to ELMO

Problem ELMO1 (Eric Shen)

Let ABC be a triangle, and let P and Q lie on sides AB and AC such that the circumcircle of $\triangle APQ$ is tangent to segment BC at a point D. Let E lie on segment BC such that $BD = EC$. Line DP intersects the circumcircle of $\triangle CDQ$ again at X, and line DQ intersects the circumcircle of $\triangle BDP$ again at Y. Prove that points D, E, X, Y are concyclic.

We present two solutions.

First solution, by angle chasing Since $\measuredangle BYD = \measuredangle BPD = \measuredangle APD = \measuredangle AQD$, we have $\overline{BY} \parallel \overline{AC}$ and analogously $\overline{CX} \parallel \overline{AB}$. Construct $A' = \overline{BY} \cap \overline{CX}$, so $ABA'C$ is a parallelogram.



Note that $\measuredangle XDY = \measuredangle PQD = \measuredangle PAQ = \measuredangle XA'Y$, so D, X, Y, A' are concyclic. Now observe that $\measuredangle A'ED = \measuredangle ADE = \measuredangle APD = \measuredangle BPD = \measuredangle BYD = \measuredangle A'YD$, so E lies on $(DXA'Y)$ as well.

Second solution, by circumcenters (Pitchayut Saengrungkongka) Let O, O_B, O_C, S be the centers of (APQ), (BDP), (CDQ), (DXY), respectively. Clearly, OO_BSO_C is parallelogram.

The rest is just projection chasing: let proj(U) denote the projection of U onto $\overline{BC}$. We have

$$\begin{aligned}\operatorname{proj}(S) &= \operatorname{proj}(O_B) + \operatorname{proj}(O_C) - \operatorname{proj}(O)\\ &= \frac{B+D}{2} + \frac{C+D}{2} - D\\ &= \frac{B+C}{2},\end{aligned}$$

so S lies on the perpendicular bisector of $\overline{BC}$, which coincides with the perpendicular bisector of $\overline{DE}$.

Problem ELMO2 (Maxim Li)

Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be integers such that $n \mid a_i - i$ for all integers $1 \leq i \leq n$. Prove there exists an infinite sequence $b_1, b_2, \ldots$ with $b_i \in \{a_1, a_2, \ldots, a_n\}$ for each i, such that

$$\sum_{i=1}^{\infty} \frac{b_i}{n^i} \in \mathbb{Z}.$$

Evidently we may select integers $z_0, z_1, z_2, \ldots$ all in $\{a_1, a_2, \ldots, a_n\}$ so that

$$z_0 + z_1 n + \cdots + z^{k-1} n^{k-1} \equiv 0 \pmod{n^k}$$

for each k. Thus each of the numbers

$$\frac{z_0}{n}, \quad \frac{z_1}{n} + \frac{z_0}{n^2}, \quad \frac{z_2}{n} + \frac{z_1}{n^2} + \frac{z_0}{n^3}, \quad \ldots$$

are integers.

Now since these integers are bounded, we may find k and ℓ with

$$\frac{z_{k-1}}{n} + \cdots + \frac{z_0}{n^k} = \frac{z_{k+\ell-1}}{n} + \cdots + \frac{z_0}{n^{k+\ell}} =: A.$$

This implies that

$$\frac{z_{k+\ell-1}}{n} + \cdots + \frac{z_k}{n^\ell} = \left(1 - \frac{1}{n^\ell}\right) A.$$

Finally, let $(b_1, b_2, \ldots) = (z_{k+\ell-1}, \ldots, z_0, z_{k+\ell-1}, \ldots, z_0, \ldots)$, so that

$$\frac{b_1}{n} + \frac{b_2}{n^2} + \cdots = \frac{\frac{z_{k+\ell-1}}{n} + \cdots + \frac{z_k}{n^\ell}}{1 - \frac{1}{n^\ell}} = \frac{z^{k-1}}{n} + \cdots + \frac{z_0}{n^\ell} = A,$$

which is an integer.

Problem ELMO3 (Maxim Li)

Each cell of a 100×100 grid is colored with one of 101 colors. A cell is *diverse* if, among the 199 cells in its row and column, every color appears at least once. Determine the maxmum possible number of diverse cells.

The answer is $100^2 - 4$, achieved by generalizing the following construction:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5\\ 3 & 1 & 2 & 4 & 5\\ 2 & 3 & 1 & 4 & 5\\ 4 & 4 & 4 & \color{red}{5} & \color{red}{6}\\ 6 & 6 & 6 & \color{red}{6} & \color{red}{5} \end{bmatrix}.$$

Now we prove that this is maximal. If some color appears at most 98 times, then we may find two rows and two columns without this color, so we may find four non-diverse squares.

Henceforth all colors appear at least 99 times, so we may assume without loss of generality the colors, 1, 2, . . . , 100, 101 each appear 99, 99, . . . , 99, 100 times.

For each $k = 1, \dots, 100$, select a color c_k that does not appear in row k, and let x_k be the number of columns containing c_k. If $x_k \neq 100$ for all k, there is a non-diverse square in every row, so at least 100 non-diverse squares.

Otherwise $x_k = 100$ for some k, implying $c_k = 101$. Thus the color 101 does not appear in every row, so repeating the argument with columns instead of rows gives the required bound.

Problem ELMO4 (Brandon Wang)

Suppose the set of positive integers is partitioned into $n \geq 2$ disjoint arithmetic progressions S_1, S_2, . . ., S_n with common differences d_1, d_2, . . ., d_n. Prove that there exists exactly one index $1 \leq i \leq n$ such that

$$\prod_{j \neq i} d_j \in S_i.$$

Evidently every sequence is a complete residue class. Exactly one sequence contains

$$\sum_i \prod_{j \neq i} d_j,$$

and that is the sequence that works.

Problem ELMO5 (Sean Li)

Let n and k be positive integers. Two infinite sequences (s_i) and (t_i) are *equivalent* if $s_i = s_j$ if and only if $t_i = t_j$ for all positive integers i and j, and a sequence (t_i) has *equi-period* k if t_1, t_2, . . . and t_{k+1}, t_{k+2}, . . . are equivalent. In terms of n and k, how many sequences of equi-period k are there in the set of sequences with each entry in the set $\{1, 2, \dots, n\}$, up to equivalence?

The answer is n^k.

We'll construct a bijection between $(a_1, \dots, a_k) \in [n]^k$ and equivalence classes of sequences (s_i) of equi-period k.

First direction: Suppose we are given $(a_1, \dots, a_k) \in [n]^k$. Define a counter $C = 0$. For each $i = 1, 2, \dots, k$:

- If $a_i \leq n - C$, then let $s_i = C + 1$, $s_{k+i} = C + 2$, $s_{2k+i} = C + 3$, . . ., $s_{(a-1)k+i} = C + a_1$, and increment C by a_i.
- Otherwise let $s_i = n+1-a_i \leq C$, which will determine s_{k+i}, s_{2k+i}, . . . through previously-known cycles using numbers $\leq C$.

Second direction: Suppose we are given (s_i). Keep a running counter $C = 0$, that we will use to number off the distinct values of observe. For each $i = 1, 2, \dots, k$:

- If all of s_i, s_{k+i}, s_{2k+i}, . . . are greater than C (i.e. we've never seen them before), then let a_i be the period of this sequence. Without loss of generality (up to equivalence) we may let $s_i = C + 1$, $s_{k+i} = C + 2$, . . ., $s_{(a_i-1)k+i} = C + a_i$, and increment C by a_i.

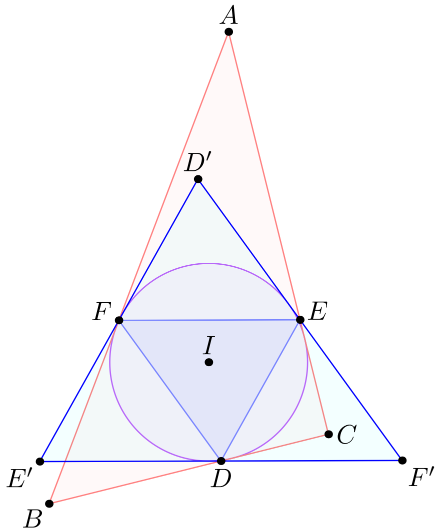
- If any of s_i, s_{k+i}, s_{2k+i}, $\dots$ is $\le C$, then all of them are determined by a previously-known cycle and each term $\le C$, so we may let $a_i = n + 1 - s_i$.

It is easy to see that the above two operations are inverses of each other, so we have established the desired bijection.

Problem ELMO6 (Maxim Li)

In triangle ABC, points D, E, F lie on segments BC, CA, AB, respectively, such that each of the quadrilaterals $AFDE$, $BDEF$, $CEFD$ has an incircle. Prove that the inradius of $\triangle ABC$ is twice the inradius of $\triangle DEF$.

Consider the anticomplementary triangle $D'E'F'$ of $\triangle DEF$. We will show that the incircles of $\triangle ABC$ and $\triangle D'E'F'$ coincide.



First note that $AF - AE = DF - DE = D'E - D'F$, so there is a circle ω_A tangent to rays AE, AF, $D'E$, $D'F$. Define ω_B and ω_C analogously.

Assume for contradiction the circles ω_A, ω_B, ω_C do not coincide. One can check that the pairwise exsimilicenters of ω_A, ω_B, ω_C are $\overline{BC} \cap \overline{E'F'} = D$, $\overline{CA} \cap \overline{F'D'} = E$, $\overline{AB} \cap \overline{D'E'} = F$. By Monge's theorem, points D, E, F are collinear, contradiction.

§2.7 Solutions to Mock IMO

Problem MIMO1 (ISL 2020 C2)

In a regular 100-gon, 41 vertices are colored black and the other 59 vertices are colored white. A quadrilateral is *weird* if it has three vertices of one color and one vertex of the other color.

Prove that there exist 24 pairwise disjoint weird quadrilaterals. (Two quadrilaterals are *disjoint* if they have no common vertices and their interiors do not intersect.)

We prove the following claim:

Claim. For $n \ge 4$, if $\#W > \#B > 0$ then we can find four consecutive vertices in the form $WWWB$ or $WWBW$.

Proof. Let the vertices have color $a_1, \ldots, a_n$ (indices modulo n) and suppose no substrings $WWWB$ or $WWBW$ occur.

Then there are no three consecutive W, since otherwise the next occurrence of B causes a $WWWB$ substring to occur. Moreover substrings WW must be immediately followed by BB, since otherwise $WWBW$ occurs.

It is easy to see, then, that $\#W \le \#B$, contradiction. □

Now take an arbitrary white vertex and delete it, so the initial value of $(\#B, \#W)$ is $(41, 58)$. Repeatedly apply the claim, removing a weird quadrilateral with three white vertices whenever $\#W > \#B$ and a weird quadrilateral with three black vertices whenever $\#B > \#W$.

Observe $\#W$ and $\#B$ always have opposite parity, so they are never equal. Moreover the ordered pair $(\#B, \#W)$ will eventually reach $(32, 31)$, and from here $|\#B - \#W| = 1$ always, so neither $\#W$ nor $\#B$ will reach 0 until there are at most three vertices remaining.

Thus we have found 24 disjoint weird quadrilaterals, as desired.

Problem MIMO2 (ISL 2020 A3)

Let a, b, c, d be positive real numbers satisfying $(a+c)(b+d) = ac+bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

By AM-GM,

$$\begin{aligned}
\left(\frac{a}{b} + \frac{c}{d}\right) + \left(\frac{b}{c} + \frac{d}{a}\right) &\ge 2\sqrt{\frac{ac}{bd}} + 2\sqrt{\frac{bd}{ac}} \\
&= 2 \cdot \frac{ac+bd}{\sqrt{ac \cdot bd}} \\
&= 2 \cdot \frac{(a+c)(b+d)}{\sqrt{ac \cdot bd}} \\
&\ge 2 \cdot \frac{2\sqrt{ac} \cdot 2\sqrt{bd}}{\sqrt{ac \cdot bd}} \\
&= 8,
\end{aligned}$$

with equality attained by $(a, b, c, d) = (2+\sqrt{3}, 1, 2+\sqrt{3}, 1)$.

Problem MIMO3 (ISL 2020 N7)

Let S be a set of $n \ge 3$ positive integers, none of which is the sum of two different numbers in S. Prove that there exists a permutation of S in which none of the middle $n-2$ integers divides the sum of its neighbors.

We present two solutions.

First solution, by explicit construction We'll construct a permutation that strictly increases, then strictly decreases.

Let $S = \{x_1 < x_2 < \cdots < x_n\}$. Then:

Claim 1. $x_i \nmid x_j + x_k$ if $i > j, k$

Proof. Note $x_j + x_k < 2x_i$. □

Claim 2. For i, j, there is at most one index $k < i$ with $x_i \mid x_j + x_k$.

Proof. If $x_i \mid x_j + x_{k_1}$ and $x_i \mid x_j + x_{k_2}$, then $x_i \mid x_{k_1} - x_{k_2}$. But $x_i > |x_{k_1} - x_{k_2}|$. $\square$

Now place x_n in our permutation, and say the initial *direction* is left. As we iterate $i = n-1, n-2, \ldots, 1$:

- If, without loss of generality, the direction is left, then try placing x_i to the left of the permutation.
- If this doesn't work, there is an "available spot" on the right of the permutation (by Claim 2), so place x_i there and set the direction to right.

This yields a valid permutation.

Second solution, by induction (mine) In what follows, the notation $x \mid y \pm z$ means $x \mid y + z$ or $x \mid y - z$, and $x \nmid y \pm z$ means $x \nmid y + z$ and $x \nmid y - z$.

We prove the following via induction on n:

> Let S be a set of n positive integers, none of which is the sum of two different numbers in S. There is a permutation of S such that each of the middle $n-2$ integers divides neither the sum nor difference of its neighbors.

Our base case is $n = 2$, for which there is nothing to show.

Now assume the permutation

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n$$

obeys the required conditions. For $A > \max\{a_1, \ldots, a_n\}$ with $A \neq a_i + a_j$ for all i, j, we will show we can insert A somewhere in $a_1, \ldots, a_n$.

First note that for all i, j we have $A \nmid a_i + a_j$. This is because $|a_i - a_j| < A$ and $a_i + a_j < 2A$ always.

Now assume for contradiction A cannot be inserted anywhere. Then the following assertions all hold:

- $a_2 \mid a_3 \pm A$ (else $a_1, A, a_2, a_3, \ldots$ works);
- $a_2 \mid a_1 \pm A$ or $a_3 \mid a_4 \pm A$ (else $a_1, a_2, A, a_3, \ldots$);
- $a_3 \mid a_2 \pm A$ or $a_4 \mid a_5 \pm A$ (else $a_1, a_2, a_3, A, \ldots$);
- $\ldots$
- $a_{n-2} \mid a_{n-3} \pm A$ or $a_{n-1} \mid a_n \pm A$ (else $\ldots, a_{n-2}, A, a_{n-1}, a_n$);
- $a_{n-1} \mid a_{n-2} \pm A$ (else $\ldots, a_{n-2}, a_{n-1}, A, a_n$);

However, note for all i that $a_i \mid a_{i-1} \pm A$ and $a_i \mid a_{i+1} \pm A$ cannot simultaneously hold, since otherwise $a_i \mid a_{i-1} \pm a_{i+1}$, contradiction. It follows that

$$\begin{aligned} a_2 \mid a_3 \pm A &\implies a_2 \nmid a_1 \pm A \implies a_3 \mid a_4 \pm A \\ &\implies a_3 \nmid a_2 \pm A \implies a_4 \mid a_5 \pm A \\ &\implies a_4 \nmid a_3 \pm A \implies a_5 \mid a_6 \pm A \\ &\implies \cdots \\ &\implies a_{n-2} \nmid a_{n-3} \pm A \implies a_{n-1} \mid a_n \pm A \\ &\implies a_{n-1} \nmid a_{n-2} \pm A, \end{aligned}$$

contradiction.

Problem MIMO4 (ISL 2020 G3)

Let $ABCD$ be a convex quadrilateral with $\min\{\angle B, \angle D\} > 90^\circ$ and $\angle A = \angle C$. Points E and F are the reflections of A in $\overline{BC}$ and $\overline{CD}$. Segments AE and AF meet line BD at K and L.

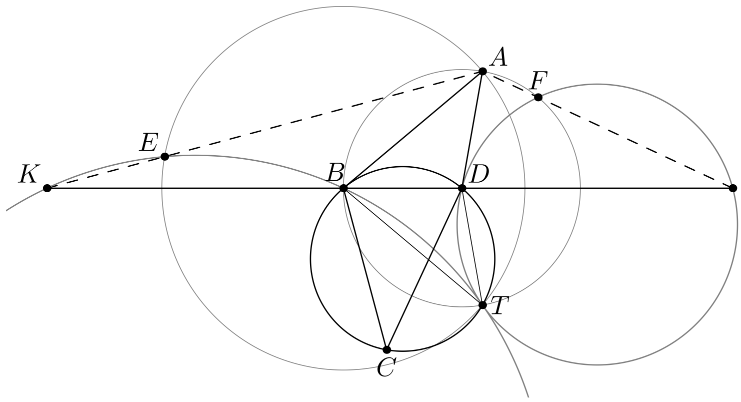
Prove that the circumcircles of $\triangle BEK$ and $\triangle DFL$ are tangent to each other.

Let T be the reflection of A over $\overline{BD}$, so that T lies on (BCD). Also note $BA = BE = BC$ and $DA = DF = DT$.

Claim. T lies on (BEK).

Proof. Note that $\overline{KB}$ bisects $\angle AKT$ and $BE = BT$. □

Analogously T lies on (DFL).



Let ℓ_1 and ℓ_2 be the tangents to $(BEKT)$ and $(DFLT)$ at T. Then note that

$$\measuredangle(\ell_1, \overline{BT}) = \measuredangle TKB = \measuredangle BKA = 90^\circ + \measuredangle DBC$$

and similarly $\measuredangle(\overline{DT}, \ell_2) = 90^\circ + \measuredangle CDB$, so

$$\begin{aligned} \measuredangle(\ell_1, \ell_2) &= \measuredangle(\ell_1, \overline{BT}) + \measuredangle BTD + \measuredangle(\overline{DT}, \ell_2) \\ &= (90^\circ + \measuredangle DBC) + \measuredangle BCD + (90^\circ + \measuredangle CDB) = 0^\circ, \end{aligned}$$

as needed.

Problem MIMO5 (ISL 2020 N5)

Determine all functions $f : \{1, 2, \dots\} \to \{0, 1, 2, \dots\}$ such that

- $f(xy) = f(x) + f(y)$ for all positive integers x and y, and
- there exists an infinite set S of positive integers such that $f(a) = f(b)$ whenever $a + b \in S$.

We present two solutions.

First solution The answer is $f(x) \equiv kv_p(x)$ for $k \geq 0$ and p prime, for which $S = \{1, p, p^2, p^3 \dots\}$ works.

Let p be minimal so that $f(p) \neq 0$. (Clearly p is prime, since if $p = ab$ then $f(a) = f(b) = 0$.)

Claim. S can only consist of numbers of the form kp^ℓ, where $\ell \ge 0$ and $k < p$.

Proof. We know $f(1), f(2), \dots, f(p-1) = 0$ and $f(pn) > 0$ for all $n \ge 1$, so if $0 < k < p$ and $n > 0$, then $np + k \notin S$; that is, elements of S are of the form

- k, with $0 < k < p$, or
- multiples of p.

However one can easily check that $ab \in S$ implies $a \in S$ and $b \in S$. Indeed, if $x + y = a$ then $xb + yb = ab$, so $f(xb) = f(yb)$, implying $f(x) = f(y)$. The claim follows. $\square$

If S is infinite, then $p^\ell \in S$ for all ℓ.

Assume that $f(q) > 0$ for some prime $q \neq p$. Then $p^{q-1} \equiv 1 \pmod q$, so $f(p^{q-1} - 1) > 0 = f(1)$, implying $p^{q-1} \notin S$, contradiction.

Second solution by magic (shortlist packet) We can directly show:

Claim. If p is any prime with $f(p) > 0$, then S can only consist of numbers of the form kp^ℓ with $k < p$.

Proof. If $s \in S$, then for any $r \le s-1$,

$$f\left(\binom{s-1}{r}\right) = \sum_{i=1}^{r} [f(s-i) - f(i)] = 0.$$

Thus $\binom{s-1}{r}$ is not divisible by p for any $r \le s-1$, so the conclusion follows from Lucas's theorem. $\square$

At last if $f(p), f(q) > 0$ then we can apply the claim to p and q, showing that S must be finite.

Problem MIMO6 (ISL 2020 C8)

Anastasia and Bananastasia play a game on a board as follows. Initially, the board contains 2020 copies of the number 1. Each round proceeds as follows:

1. Anastasia erases two numbers x and y from the board.
2. Bananastasia writes one of $x + y$ and $|x - y|$ on the board.

After each round, the game ends if one of the following holds:

- one number on the board is larger than the sum of all other numbers on the board, or
- all numbers on the board are zeroes.

After the game ends, Bananastasia must give Anastasia one slice of banana bread for every number remaining on the board. How many slices of banana bread can Anastasia guarantee, assuming optimal play from both players?

The answer is 7; for general n, the answer is $s_2(n)$, the sum of the digits of n in binary.

Anastasia's strategy: We will show Ana can guarantee $s_2(n)$ numbers remain for all n. To this end, we induct on n, letting $f(n)$ denote the answer for n.

Pair up the 1's (potentially leaving a 1), and ask Banana to combine them. In the end, there are m twos, $\lfloor n/2 \rfloor - m$ zeros, and $n \bmod 2$ ones.

Then Ana focuses on the m twos, repeating her strategy. It can be seen that when the game on the m twos terminates, the original game is also over. Hence the number of terms remaining is

$$\begin{aligned} f(n) &\geq (n \bmod 2) + \min_m (f(m) + \lfloor n/2 \rfloor - m) \\ &\geq (n \bmod 2) + \min_m (s_2(m) + \lfloor n/2 \rfloor - m) \\ &= (n \bmod 2) + s_2(\lfloor n/2 \rfloor) = s_2(n). \end{aligned}$$

Bananastasia's strategy: Let n be even. In essence, a strategy for Ana is a binary tree of possibilities for Banana, given the current configuration of the board.

For each node corresponding to position $a_1, \ldots, a_n$, consider the multiset

$$S = \{\pm a_1 \pm a_2 \pm \cdots \pm a_n\}$$

of size 2^n.

Claim (Black magic)**.** For each node of the binary tree with corresponding multiset S, if its children have multisets S_1 and S_2 then $S = S_1 \sqcup S_2$.

Proof. If Ana's strategy chooses a and b, then the four values of $\pm a \pm b$ match the two values of $\pm(a+b)$ and the two values of $\pm(a-b)$. $\square$

For such a binary tree, by taking the disjoint union of the multisets of all the leaves, you get the original multiset.

There are $\binom{n}{n/2}$ zeroes in the root's multiset. If we consider the leaves that correspond to terminated configurations,

- the multisets of those in which one number is larger than the sum of the rest have no zeroes, and
- the multisets (with size m) of those in which all numbers are zeroes have 2^m zeroes.

It follows that

$$\binom{n}{n/2} = \sum 2^m,$$

implying $\min m \leq \nu_2(\binom{n}{n/2}) = s_2(n)$.

Remark (Explicit strategy)**.** We may explicitly state Banana's strategy as follows. If F is the number of zeroes in S for some given configuration, then Banana's strategy is to ensure that $\nu_2(F) \leq s_2(n)$ always. This is always possible, since

- the initial value of F, i.e. $\binom{n}{n/2}$, has this property, and
- if the two possible values of F after Banana's move are F_1 and F_2, the claim implies $F = F_1 + F_2$.

Remark (n odd)**.** The shortlist packet shows it is possible to modify the above remark to solve the problem for n odd.

We instead keep track of the sets

$$\begin{aligned} S_1 &= \{+a_1 \pm a_2 \pm a_3 \pm \cdots \pm a_n\} \\ S_2 &= \{\pm a_1 + a_2 \pm a_3 \pm \cdots \pm a_n\} \\ S_3 &= \{\pm a_1 \pm a_2 + a_3 \pm \cdots \pm a_n\} \\ &\vdots \\ S_n &= \{\pm a_1 \pm a_2 \pm a_3 \pm \cdots + a_n\}. \end{aligned}$$

Let F_i be the number of positive terms in S_i, and let $F = \min\{F_i\}$. The initial value of F is

$$2^{n-1} + \frac{1}{2}\binom{n-1}{(n-1)/2},$$

with $\nu_2(F) = s_2(n) - 2$.

If F splits into F_1 and F_2 then it can be seen that $\min\{\nu_2(F_1), \nu_2(F_2)\} \le \nu_2(F)$, so Banana can guarantee $\nu_2(F) \le s_2(n) - 2$ always.

The multisets (with size m) of the terminated leaves, i.e. those in which one number is larger than the rest, will have $F = 2^{m-2}$, and the desired conclusion follows.

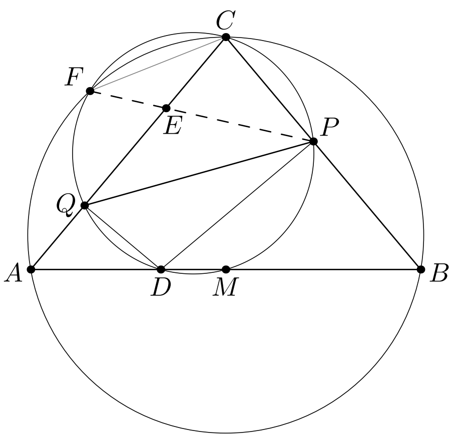
§2.8 Solutions to MOP Quiz 6

Problem B6.1 (ISL 2020 G1)

Let ABC be an isosceles triangle with $CA = CB$, and let D be a point on side AB with $AD < DB$. Let P and Q be the projections from D to $\overline{CB}$ and $\overline{CA}$. The perpendicular bisector of $\overline{PQ}$ meets segment CQ at E, and the circumcircles of $\triangle ABC$ and $\triangle CPQ$ meet at $F \neq C$.

Show that if P, E, F are collinear, then $\angle C = 90^\circ$.

Let M be the midpoint of $\overline{AB}$, so $CPMDQF$ is cyclic. Since $PCFQ$ is an isosceles trapezoid and $MP = MQ$ (from $\angle QCM = \measuredangle MCP$), we have $MC = MF$. Drawing the circle centered at M through C, it is clear $MC = MF$ is only possible if M is the center of (ABC).



Problem B6.2 (ISL 2020 A1)

Let n be a positive integer. Determine the smallest real number C such that, for all real x,

$$\sqrt[n]{\frac{x^{2n}+1}{2}} \le C(x-1)^2 + x.$$

The answer is $C \geq n/2$.

If $f(x) = \left(\frac{x^{2n}+1}{2}\right)^{1/n}$ and $g(x) = C(x-1)^2 + x$, we may compute that

$$\begin{aligned}
f'(x) &= x^{2n-1}\left(\frac{x^{2n}+1}{2}\right)^{1/n-1} \\
f''(x) &= \frac{1}{2}x^{2n-2}\left(x^{2n}+2n-1\right)\left(\frac{x^{2n}+1}{2}\right)^{1/n-2} \\
f'''(x) &= \frac{-(n-1)(2n-1)}{2}\cdot x^{2n-3}\left(x^{2n}-1\right)\left(\frac{x^{2}+1}{2}\right)^{1/n-3}.
\end{aligned}$$

Moreover

$$g'(x) = 2C\cdot x - (2C-1) \quad \text{and} \quad g''(x) = 2C.$$

Proof $c = n/2$ works: Analysis of f''' shows that the local maxima of f'' are at 1 and -1. Therefore we have

$$f''(x) \leq g''(x)\ \forall x.$$

Since $f'(1) = g'(1) = 1$, we have

$$\begin{aligned}
f'(x) &\geq g'(x)\ \forall x \leq 1, \\
f'(x) &\leq g'(x)\ \forall x \geq 1.
\end{aligned}$$

Finally since $f(1) = g(1) = 1$, we have

$$f(x) \leq g(x)\ \forall x.$$

Proof $c < n/2$ fails: We have that $f''(1) = n > 2c = g''(1)$, so by continuity for some $\varepsilon > 0$ we have

$$f''(x) > g''(x)\ \forall x \in (1-\varepsilon, 1+\varepsilon).$$

Since $f'(1) = g'(1) = 1$, we have

$$\begin{aligned}
f'(x) &< g'(x)\ \forall x \in (1-\varepsilon, 1) \\
f'(x) &> g'(x)\ \forall x \in (1, 1+\varepsilon).
\end{aligned}$$

Finally since $f(1) = g(1) = 1$, we have

$$f(x) > g(x)\ \forall x \in (1-\varepsilon, 1+\varepsilon) \setminus \{1\},$$

contradiction.

> **Remark.** You can be almost sure of the answer by computing the Taylor approximation
>
> $$\sqrt[n]{\frac{x^{2n}+1}{2}} \approx 1 + (x-1) + n\cdot\frac{(x-1)^2}{2}.$$

Remark (Alternate solution sketch, Ankan Bhattacharya)**.** Let $x = \frac{1+\varepsilon}{1-\varepsilon}$ for $\varepsilon \in \mathbb{R}$. Then we want to show

$$\begin{aligned} (1+\varepsilon)^{2n} + (1-\varepsilon)^{2n} &\le \left(\frac{n}{2}(2\varepsilon)^2 + (1-\varepsilon)(1+\varepsilon)\right)^n \\ \iff (1-\varepsilon)^{2n} + (1+\varepsilon)^{2n} &\le \left(1 + D\varepsilon^2\right)^n, \end{aligned}$$

where it turns out $D = 2n-1$, by checking that every coefficient on the left is smaller than the corresponding coefficient on the right.

Remark. Mildorf once proved the following in his quintessential handout about inequalities:

> Let $k \ge -1$ be an integer. Then for all positive reals a and b,
>
> $$\frac{(1+k)(a-b)^2 + 8ab}{4(a+b)} \ge \sqrt[k]{\frac{a^k+b^k}{2}}$$
>
> with equality if and only if $a = b$ or $k = \pm 1$, where the power mean $k = 0$ is interpreted to be the geometric mean $\sqrt{ab}$. Moreover, if $k < -1$, then the inequality holds in the reverse direction, with equality if and only if $a = b$.

§2.9 Solutions to MOP Test 7

Problem B7.1 (ISL 2020 G4)

Let $n \ge 6$ be an integer and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be pairwise disjoint closed disks in the plane with radii $R_1 \ge \cdots \ge R_n$. For each $i \in \{1, \ldots, n\}$, let P_i be a point on $\mathcal{D}_i$. Let O be a point in the plane. Prove that

$$OP_1 + OP_2 + \cdots + OP_n \ge R_6 + R_7 + \cdots + R_n.$$

I contend $OP_i \ge R_6$ for some $i \le 6$. This suffices by induction.

Drop the condition $R_1 \ge R_2 \ge \cdots \ge R_6$, and instead number the disks $\mathcal{D}_1, \ldots, \mathcal{D}_6$ counter-clockwise with respect to O. Let the center of $\mathcal{D}_i$ be O_i.

Since $\angle O_1OO_2 + \cdots + \angle O_6OO_1 = 360°$, for some index i we have $\angle O_iOO_{i+1} \le 60°$. Without loss of generality $OO_i \ge OO_{i+1}$. Evidently $\angle O_iOO_{i+1}$ is not the largest angle in $\triangle O_iOO_{i+1}$, i.e. $\overline{O_iO_{i+1}}$ is not the longest side, so

$$OO_i \ge O_iO_{i+1} \ge R_i + R_{i+1}.$$

This allows us to conclude

$$OP_i \ge OO_i - R_i \ge R_{i+1} \ge \min\{R_1, \ldots, R_6\}.$$

Problem B7.2 (ISL 2020 C5)

Let p be an odd prime, let $N = \frac{1}{4}(p^3 - p) - 1$, and let S be a subset of $\{1, \ldots, N\}$. Show that there exists an integer $a \in \{1, \ldots, p-1\}$ such that for all positive integers $n \in N$,

$$\frac{|S \cap \{1, \ldots, n\}|}{n} \ne \frac{a}{p}.$$

Let $S_n = |S \cap \{1, \ldots, n\}|$.

Assume for contradiction there exist $n_1 < n_2 < \cdots < n_{p-1}$ and a permutation $a_1, \ldots, a_{p-1}$ of $\{1, \ldots, p-1\}$ such that $S_{pn_i} = a_i n_i$. The goal is to show $n_{p-1} \geq \frac{p^2-1}{4}$.

Drop the p prime condition. Analyzing the difference between S_i and S_{i+1}, we must have

$$0 \leq n_{i+1}a_{i+1} - n_i a_i \leq n_{i+1}p - n_i p.$$

In other words,

$$n_{i+1} \geq n_i \cdot \max\left\{\frac{a_i}{a_{i+1}}, \frac{p-a_i}{p-a_{i+1}}\right\}.$$

The above condition suffices to prove the problem.

The idea is that we will go "up" to $p-1$ and then go "down." Let k be the largest integer such that $1, \ldots, k$ all appear before $a_v = p-1$.

Case 1: $k \leq \frac{p-1}{2}$. Let $a_u = \ell$ be the last of $1, \ldots, k$ that appear, and let $a_w = k+1$. (Hence we have $1 \leq u < v < w \leq p-1$.)

But (n_i) is strictly increasing, so $L \geq 1$, $n_u \geq u \geq k$.

$$\begin{aligned} n_{p-1} \geq n_w &\geq \max\left\{\frac{a_v}{a_w}, \frac{p-a_v}{p-a_w}\right\} c_v \\ &\geq \max\left\{\frac{a_v}{a_w}, \frac{p-a_v}{p-a_w}\right\} \max\left\{\frac{a_u}{a_v}, \frac{p-a_u}{p-a_v}\right\} n_u \\ &\geq \frac{p-1}{k+1} \cdot (p-\ell) \cdot k \geq \frac{p-1}{k+1} \cdot (p-k) \cdot k \geq \frac{p^2-1}{4} \end{aligned}$$

Case 2: $k > \frac{p-1}{2}$. In this case, jumping up suffices. Let $a_u = \ell$ be the last of $1, \ldots, \frac{p-1}{2}$ to appear. Then

$$\begin{aligned} n_{p-1} \geq n_v &\geq \max\left\{\frac{a_u}{a_v}, \frac{p-a_u}{p-a_v}\right\} n_u \\ &= (p-\ell) \cdot n_u \geq \frac{p+1}{2} \cdot \frac{p-1}{2} = \frac{p^2-1}{4}. \end{aligned}$$

Problem B7.3 (ISL 2020 G8)

Let ABC be a triangle with incenter I and circumcircle Γ. Circles ω_B passing through B and ω_C passing through C are tangent at I. Let ω_B meet minor arc AB of Γ at P and $\overline{AB}$ at $M \neq B$, and let ω_C meet minor arc AC of Γ at Q and $\overline{AC}$ at $N \neq C$, Rays PM and QN meet at X. Let Y be a point such that $\overline{YB}$ is tangent to ω_B and $\overline{YC}$ is tangent to ω_C.

Show that A, X, Y are collinear.

Let $\overline{PM}$ and $\overline{QN}$ intersect Γ again at P' and Q'.

Claim 1. *$BCNM$ is tangential, and $Y \in \Gamma$.*

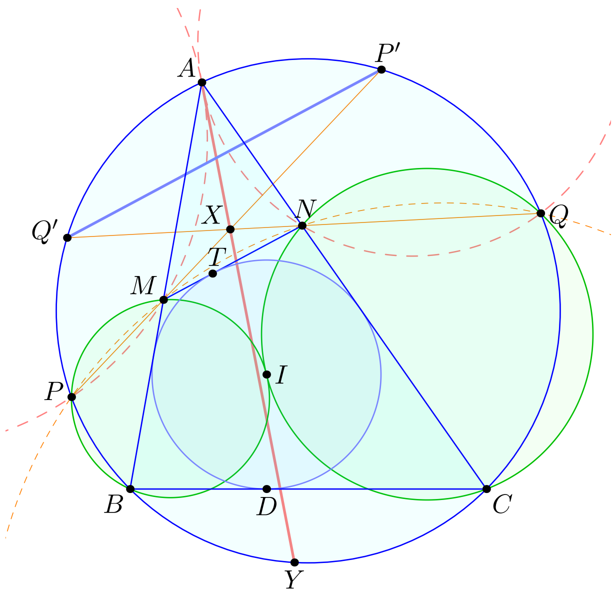
Proof. If ℓ is the common tangent to ω_B and ω_C at I, then

$$\begin{aligned} \measuredangle MIN &= \measuredangle(\overline{MI}, \ell) + \measuredangle(\ell, \overline{NI}) \\ &= \measuredangle MBI + \measuredangle ICN = \measuredangle IBC + \measuredangle BCI = \measuredangle BIC, \end{aligned}$$

implying that $BCNM$ is tangential. Moreover

$$\measuredangle YBM = \measuredangle BIM = \measuredangle CIN = \measuredangle YCN,$$

so Y lies on Γ. $\square$



Claim 2. $\overline{YA}$ is tangent to (AMP) and (ANQ).

Proof. Note that

$$\measuredangle APM = \measuredangle APB + \measuredangle BPM = \measuredangle AYB + \measuredangle YBM = \measuredangle YAM,$$

and analogously $\measuredangle AQN = \measuredangle YAN$. $\square$

Claim 3. M, N, P, Q are concyclic.

Proof. If $\overline{PM}$ and $\overline{QN}$ intersect Γ again at P' and Q', then By Reim's theorem it will suffice to show $\overline{MN} \parallel \overline{P'Q'}$.

If the incircle touches $\overline{BC}$ and $\overline{MN}$ at D and T, then

$$\begin{aligned} \measuredangle(\overline{P'Q'}, \overline{BC}) &= \measuredangle Q'P'C + \measuredangle P'CB = \measuredangle NQC + \measuredangle MPB \\ &= \measuredangle NIC + \measuredangle MIB = 2\measuredangle MIB = \measuredangle TID = \measuredangle(\overline{MN}, \overline{BC}), \end{aligned}$$

as required. $\square$

By radical axis theorem on (AMP), (ANQ), $(MNPQ)$, the desired conclusion follows, with points A, P, Q on the common tangent of (AMP) and (ANQ).

Remark (Alternate solution sketch)**.** After finding $Y \in \Gamma$, we may invert at I. It turns out $MNPQ$ becomes an isosceles trapezoid, and it is not hard to finish from here, either in the inverted or original diagram.