USAJMO 2019

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§0 Problems

Problem 1. There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear.

A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

Problem 2. Let $\mathbb{Z}$ be the set of all integers. Find all pairs of integers (a,b) for which there exist functions $f:\mathbb{Z}\to\mathbb{Z}$ and $g:\mathbb{Z}\to\mathbb{Z}$ satisfying

$$f(g(x)) = x+a \quad \text{and} \quad g(f(x)) = x+b$$

Problem 3. Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2+BC^2=AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side AB satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

Problem 4. Let ABC be a triangle with $\angle ABC$ obtuse. The A-excircle is a circle in the exterior of $\triangle ABC$ that is tangent to side BC of the triangle and tangent to the extensions of the other two sides. Let E, F be the feet of the altitudes from B and C to lines AC and AB, respectively. Can line EF be tangent to the A-excircle?

Problem 5. Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j}\subseteq\{1,2,\dots,2n\}$, for integers $0\le i\le n$ and $0\le j\le n$ such that:

- for all $0\le i,j\le n$, the set $S_{i,j}$ has $i+j$ elements; and
- $S_{i,j}\subseteq S_{k,l}$ whenever $0\le i\le k\le n$ and $0\le j\le l\le n$.

Problem 6. Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m,n) such that Evan can write 1 on the board in finitely many steps.

§1 USAJMO 2019/1 (Jim Propp)

Problem 1 (USAJMO 2019/1)

There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear.

A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

The problem consists of two parts:

Proof of sufficiency: Assume ab is even, and since the problem is symmetric under reflection, without loss of generality let a be even. I will show by induction on b that the goal is always possible.

Consider $b=0$ as the base case — there is nothing to show. Take the leftmost pear (in bowl $a+1$), and move it all the way to bowl 1; when the pear moves from bowl n to bowl $n-1$, also move the apple in bowl $a+2-n$.

Each apple moves exactly once, and so after this process, the leftmost bowl contains a pear, the next a bowls each contain an apple, and the rightmost $b-1$ bowls contain a pear. Then apply the induction hypothesis on $b-1$ to sort the remaining $a+b-1$ apples.

Proof of necessity: Assume ab is odd. We can check that in total, ab moves occur. There are two types of moves:

- Moving two fruit from odd-numbered bowls to even-numbered bowls.
- Moving two fruit from even-numbered bowls to odd-numbered bowls.

Since in both the beginning and the end of the process, each bowl contains a fruit, the two types of moves occur equally often. This is absurd, since ab is odd.

§2 USAJMO 2019/2 (Ankan Bhattacharya)

Problem 2 (USAJMO 2019/2)

Let $\mathbb{Z}$ be the set of all integers. Find all pairs of integers (a, b) for which there exist functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(g(x)) = x + a \quad \text{and} \quad g(f(x)) = x + b$$

The answer is $|a| = |b|$. Here are possible constructions:

- For (a, a), take $f(x) \equiv x + 2a$, $g(x) \equiv x - a$.
- For $(a, -a)$, take $f(x) \equiv 2a - x$, $g(x) \equiv a - x$.

Evidently $a = 0 \iff b = 0$, so in what follows, we assume a, b are nonzero.

Assume for contradiction $|a| \neq |b|$, and without loss of generality $|a| > |b|$. By the triple involution trick,

$$f(x + b) = f(g(f(x))) = f(x) + a$$

Hence all residues of $f(x)$ modulo $|a|$ are covered by $f(0)$, $f(1)$, ..., $f(|b| - 1)$; in particular, f is not surjective modulo $|a|$. This is absurd, since $f(g(x))$ is clearly surjective.

§3 USAJMO 2019/3 (Ankan Bhattacharya)

Problem 3 (USAMO 2019/2)

Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side AB satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

First solution, by symmedians There is a point P on side AB with $AP = AD^2/AB$ and $BP = BC^2/AB$. We have

$$\frac{PA}{PB} = \left(\frac{AD}{BC}\right)^2 = \left(\frac{EA}{EB}\right)^2$$

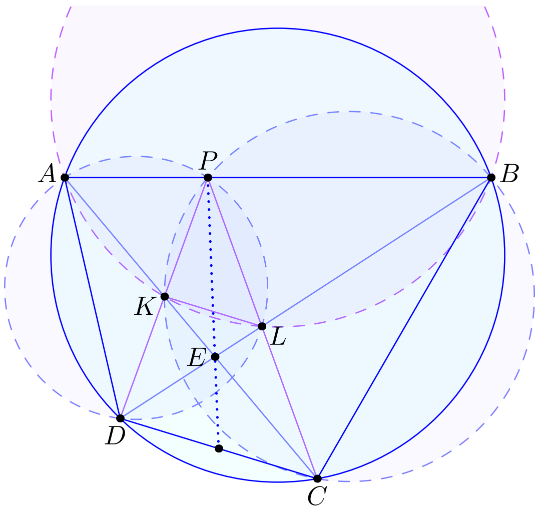
since $\triangle AED \sim \triangle BEC$. Then $\overline{EP}$ is the E-symmedian of $\triangle EAB$, so $\overline{EP}$ bisects $\overline{CD}$.

By monotonicity it remains to check $\angle APD = \angle BPC$. Since $AP \cdot AB = AD^2$, we have $\triangle APD \sim \triangle ADB$, and analogously $\triangle BPC \sim \triangle BCA$. Hence $\measuredangle APD = \measuredangle BDA = \measuredangle BCA = \measuredangle CPB$, end proof.

Second (elementary) solution, from official solutions packet As before, construct P so that $AP \cdot AB = AD^2$ and $BP \cdot BA = BC^2$. Then $\triangle APD \sim \triangle ADB$, $\triangle BPC \sim \triangle BCA$, so

$$\theta := \measuredangle APD = \measuredangle BDA = \measuredangle BCA = \measuredangle CPB.$$

The task is to show $\overline{PE}$ bisects $\overline{CD}$.



Let $K = \overline{AC} \cap \overline{PD}$, $L = \overline{BD} \cap \overline{PC}$.

Claim 1. $APLD$ and $BPKC$ are cyclic.

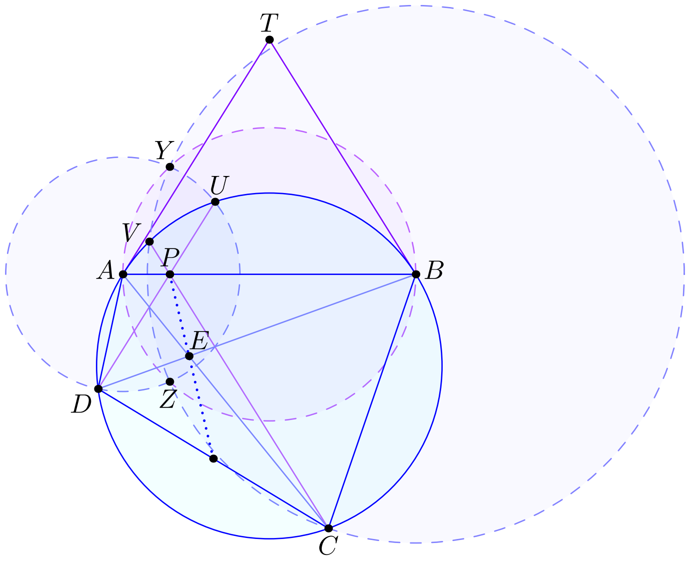
Proof. These follow from $\measuredangle LDA = \theta = \measuredangle LPA$ and $\measuredangle BCK = \theta = \measuredangle BPK$ respectively. $\square$

Claim 2. $AKLB$ is cyclic.

Proof. We have $\measuredangle AKB = \measuredangle CKB = \measuredangle CPB = \theta$ and similarly $\measuredangle ALB = \theta$. $\square$

By Reim's theorem on $(ABCD)$, $(AKLB)$, we have $\overline{KL} \parallel \overline{CD}$. If $M = \overline{PE} \cap \overline{CD}$, then $MC = MD$ by Ceva's theorem on $\triangle PCD$.

Third solution, by radical axes Let Ω be the circumcircle of $ABCD$, let Γ be the circle with diameter AB, and let ω_A, ω_B be the circles centered at A, B with radii AD, BC, respectively.



The condition $AD^2 + BC^2 = AB^2$ means ω_A, ω_B are orthogonal, so their intersection points Y, Z lie on Γ. Let P be the midpoint of $\overline{YZ}$. Then P lies on $\overline{AB}$, the radical axis of Ω, Γ, so P is the common radical center of Ω, Γ, ω_A, ω_B.

Let the tangents to Ω at A, B meet at T, and let $\overline{DP}$, $\overline{CP}$ meet Ω again at U, V. Since $\overline{DU}$ is the radical axis of Ω, ω_A, we have $\overline{AT} \parallel \overline{DU}$. Similarly $\overline{BT} \parallel \overline{CV}$, so $\measuredangle APD = \measuredangle BAT = \measuredangle TBA = \measuredangle CPB$. By monotonicity, P is the point described in the problem statement.

Finally we have

$$\frac{PA}{PB} = \left(\frac{YA}{YB}\right)^2 = \left(\frac{AD}{BC}\right)^2 = \left(\frac{EA}{EB}\right)^2,$$

so $\overline{EP}$ is the E-symmedian of $\triangle EAB$, which bisects $\overline{CD}$.

> **Remark.** This was my solution in-contest, and I believe it is pretty motivated. The four circles in the problem are the most natural way to construct the diagram, and an in-scale diagram suggests P lies on $\overline{YZ}$. The rest is not hard to prove.

Fourth solution, by inversion (Evan Chen) As noted above, the circle ω_A centered at A with radius AD is orthogonal to the circle ω_B centered at B with radius BC. We let $\mathbf{I}_A$, $\mathbf{I}_B$ denote inversion with respect to ω_A, ω_B.

Let the radical axis of ω_A, ω_B intersect $\overline{AB}$ at P; by design, $P = \mathbf{I}_A(B) = \mathbf{I}_B(A)$. This already implies that

$$\measuredangle APD \overset{\mathbf{I}_A}{=} \measuredangle BDA = \measuredangle BCA \overset{\mathbf{I}_B}{=} \measuredangle CPB,$$

so by monotonicity, P is the point described in the problem statement.

> **Claim.** The point $K = \mathbf{I}_A(C)$ lies on ω_B and $\overline{DP}$. Similarly $L = \mathbf{I}_B(D)$ lies on ω_A and $\overline{CP}$.

Proof. Since $\omega_A \perp \omega_B$, it follows that $K \in \omega_B$. For $K \in \overline{DP}$, note $ABCD$ is cyclic, so $P = \mathbf{I}_A(B)$, $K = \mathbf{I}_A(C)$, $D = \mathbf{I}_A(D)$ are collinear. $\square$

Finally since C, L, P collinear, A is concyclic with $K = \mathbf{I}_A(C)$, $L = \mathbf{I}_A(L)$, $B = \mathbf{I}_A(B)$, i.e. $AKLB$ is cyclic. Hence $\overline{KL} \parallel \overline{CD}$ by Reim's theorem, and $\overline{PE}$ bisects $\overline{CD}$ by Ceva's theorem.

§4 USAJMO 2019/4 (Ankan Bhattacharya, Zack Chroman, Anant Mudgal)

Problem 4 (USAJMO 2019/4)

Let ABC be a triangle with $\angle ABC$ obtuse. The A-excircle is a circle in the exterior of $\triangle ABC$ that is tangent to side BC of the triangle and tangent to the extensions of the other two sides. Let E, F be the feet of the altitudes from B and C to lines AC and AB, respectively. Can line EF be tangent to the A-excircle?

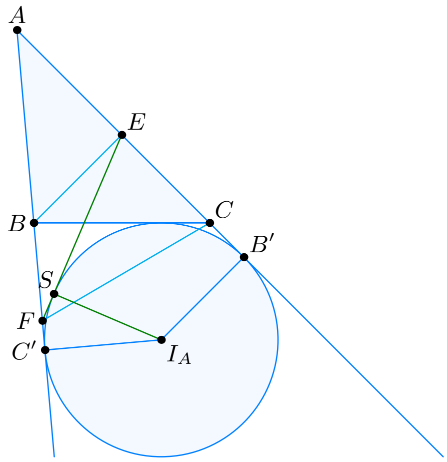
It is not possible. Assume for contradiction line EF is tangent to the A-excircle.

First solution, by similarity Since $\angle B = 90°$, we have A, E, C collinear in that order, and also A, B, F collinear in that order.

Note the A-excircle is tangent to all three sides of $\triangle AEF$; I contend it is its A-excircle as well. Indeed, the A-excircle lies in the interior of $\angle EAF$, but it touches $\overline{AC}$ farther away from A than E.

But $\triangle ABC$, $\triangle AEF$ are inversely similar, and they share an A-exradius, so they are congruent. Now $AB = AE = AB\cos A$, so $\cos A = 1$ and $\angle A = 0°$, absurd.

Second solution, by length chasing Let the A-excircle touch $\overline{AC}$, $\overline{AB}$, $\overline{EF}$ at B', S, C'.



If ρ denotes the semiperimeter, then

$$\begin{aligned} BC\cos A = EF &= ES + FS = EB' + FC' \\ &= (\rho - AE) + (\rho - AF) \\ &= AB + BC + CA - (AB + CA)\cos A, \end{aligned}$$

from which $\cos A = 1$, absurd.

§5 USAJMO 2019/5 (Ricky Liu)

Problem 5 (USAMO 2019/4)

Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \dots, 2n\}$, for integers $0 \le i \le n$ and $0 \le j \le n$ such that:

- for all $0 \le i, j \le n$, the set $S_{i,j}$ has $i + j$ elements; and
- $S_{i,j} \subseteq S_{k,l}$ whenever $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

The answer is $(2n)! \cdot 2^{n^2}$.

First solution, by direct computation Place the sets in an $(n+1) \times (n+1)$ grid shown below, and select the blue sets $S_{0,0}$, $S_{0,1}$, $\dots$, $S_{0,n}$, $S_{1,n}$, $\dots$, $S_{n,n}$. Each admits exactly one more element than the previous set in the sequence, so there are $(2n)!$ ways to select the blue sets.

$$\begin{bmatrix} 1357 & 12357 & 123457 & 1234578 & 12345678 \\ 135 & 1345 & 13457 & 134578 & 1345678 \\ 13 & 135 & 1357 & 13578 & 135678 \\ 1 & 15 & 157 & 1578 & 15678 \\ \varnothing & 5 & 57 & 578 & 5678 \end{bmatrix}$$

It remains to select the remaining sets. First note that the conditions given in the problem statement are equivalent to $S_{i,j} \subset S_{i+1,j}$ and $S_{i,j} \subset S_{i,j+1}$ for all $0 \le i, j \le n$ (by transitivity).

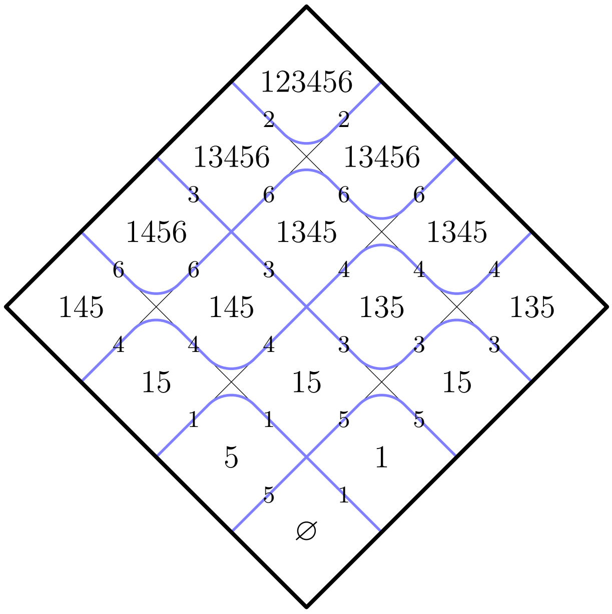
We determine the remaining sets in the following order: $S_{1,n-1}$, $S_{1,n-2}$, $\dots$, $S_{1,0}$, $S_{2,n-1}$, $S_{2,n-2}$, $\dots$, $S_{2,0}$, and so on. When it comes to determine $S_{i,j}$, we will have already chosen $S_{i-1,j}$ and $S_{i,j+1}$.

$$\begin{bmatrix} 1357 & 12357 \\ 135 & 1345 \end{bmatrix}$$

It is given $S_{i,j} \setminus S_{i-1,j}$ is a single-element subset of $S_{i,j+1} \setminus S_{i-1,j}$, which has cardinality 2; hence we have 2 choices.

There were $(2n)!$ ways to select the initially blue sets, and 2^{n^2} ways to select the remaining sets. The conclusion follows.

Second solution, by bijection (Daniel Zhu) Arrange the sets into the obvious $(n+1) \times (n+1)$ square grid. Here, we orient the grid diagonally so that $S_{0,0}$ is on the bottom and $S_{n,n}$ is on the top. Label an internal edge with i if crossing an edge while going up adds the element i to the set in the relevant square.



It follows from the $S_{i,j} \subseteq S_{k,\ell}$ condition that the set of edges with label i must follow a path from the left end of the square to the right end of the square, so we can associate every valid arrangement of sets with a partitioning of the internal edges into $2n$ paths, labeled 1, 2, $\ldots$, $2n$, traveling from the left end of the square to the right end of the square. Indeed, this correspondence is reversible, since given the system of labeled paths one can associate a square with the set of all numbers k so that the path labeled with k passes below the square.

It suffices to show that there are 2^{n^2} ways to choose all the ways to partition the unlabeled edges into $2n$ unlabeled paths. However, since every edge is utilized, this is equivalent to, for each of the n^2 internal vertices, choosing whether the two paths that pass through the vertex end up crossing or not. Thus, we are done.

§6 USAJMO 2019/6 (Yannick Yao)

Problem 6 (USAMO 2019/5)

Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m,n) such that Evan can write 1 on the board in finitely many steps.

This is possible if and only if $m+n$ is a power of 2. In what follows, let $r=\frac{m}{n}$, so we begin with r and $\frac{1}{r}$ on the board.

Proof of sufficiency: Assume $m+n$ is a power of 2. It is possible to write 1 on the board only by taking arithmetic means.

Claim 1. If x and y are on the board, then for all nonnegative integers u, v, t with $u+v=2^t$, it is possible to construct

$$\frac{u}{2^t}\cdot x+\frac{v}{2^t}\cdot y.$$

Proof. The proof is by induction on t, with $t=0$ given. Now suppose the hypothesis holds for t; we will show it holds for $t+1$.

For $u+v=2^{t+1}$, if $0\in\{u,v\}$, then we are already done. Otherwise, note that

$$\frac{u}{2^{t+1}}\cdot x+\frac{v}{2^{t+1}}\cdot y=\frac{1}{2}\left[\left(\frac{1}{2^t}\cdot x+\frac{1}{2^t}\cdot y\right)+\left(\frac{u-1}{2^t}\cdot x+\frac{v-1}{2^t}\cdot y\right)\right],$$

which is constructable. □

Finally take $x=r$, $y=\frac{1}{r}$, $u=n$, $v=m$. We can construct

$$\frac{n}{m+n}\cdot r+\frac{m}{m+n}\cdot\frac{1}{r}=1.$$

Proof of necessity: Suppose p is an odd prime dividing $m+n$. Then both numbers on the board are equivalent to $-1 \pmod p$. The main idea is in the following claim:

Claim 2. If $x\equiv y\equiv -1 \pmod p$, then both their arithmetic mean and harmonic mean are $-1 \pmod p$.

Proof. This is fairly obvious: since $2\not\equiv 0 \pmod p$ and $x+y\equiv -2\not\equiv 0 \pmod p$, we can check

$$\frac{x+y}{2}\equiv\frac{-2}{2}\equiv -1 \pmod p \quad\text{and}\quad \frac{2xy}{x+y}\equiv\frac{2}{\frac{1}{x}+\frac{1}{y}}\equiv\frac{2}{-2}\equiv -1 \pmod p,$$

and the claim follows. □

If Evan can write 1 on the board, then $1\equiv -1 \pmod p$, which is absurd.