USAJMO 2018

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§0 Problems

Problem 1. For each positive integer n, find the number of n-digit positive integers that satisfy both of the following conditions:

- no two consecutive digits are equal; and
- the last digit is a prime.

Problem 2. Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2)\geq a^2+b^2+c^2.$$

Problem 3. Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC}\perp\overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ=FR$.

Problem 4. Find all real numbers x such that there is a triangle ABC with $\angle ABC\geq 90^\circ$ inscribed in a circle of radius 2, such that if $a=BC$, $b=CA$, $c=AB$, then $x^4+ax^3+bx^2+cx+1=0$.

Problem 5. Let p be a prime, and let $a_1,\dots,a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1+k,\ a_2+2k,\ \dots,\ a_p+pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

Problem 6. Karl starts with n cards labeled $1,2,3,\dots,n$ lined up in a random order on his desk. He calls a pair (a,b) of these cards swapped if $a>b$ and the card labeled a is to the left of the card labeled b.

Karl picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on until he has picked up and put back each of the cards 1, 2, …, n exactly once in that order.

For example, one such process for $n=4$ is

$$3142\to 3412\to 2341\to 2431\to 2341.$$

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

§1 USAJMO 2018/1 (Zachary Franco, Zuming Feng)

Problem 1 (USAJMO 2018/1)
For each positive integer n, find the number of n-digit positive integers that satisfy both of the following conditions:

- no two consecutive digits are equal; and
- the last digit is a prime.

Let a_n denote the answer for n.

Claim. For all $n > 1$, $a_n = 4 \cdot 9^{n-1} - a_{n-1}$

Proof. We can check $4 \cdot 9^{n-1} = a_n + a_{n-1}$, by writing a leading zero in front of each $(n-1)$-digit number. $\square$

Solving the recursion with $a_1 = 4$ gives

$$a_n = 4 \cdot 9^{n-1} - 4 \cdot 9^{n-2} + \cdots + (-1)^{n-1} \cdot 4 = \frac{2}{5}\left(9^n - (-1)^n\right),$$

as desired.

§2 USAJMO 2018/2 (Titu Andreescu)

Problem 2 (USAMO 2018/1)
Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2)\geq a^2+b^2+c^2.$$

Without loss of generality $a\leq b\leq c$, and by homogeneity let $abc=1$, i.e. $a+b+c=4$. Then observe the following, where the first inequality follows from AM-GM:

$$\begin{aligned} 4a+\tfrac{1}{a} &\geq 4 \\ \implies a(a+b+c)+bc &\geq 4 \\ \implies 4(a^2+ab+bc+ca) &\geq (a+b+c)^2 \\ \implies 2(ab+bc+ca)+4a^2 &\geq a^2+b^2+c^2, \end{aligned}$$

Remark. Equality holds when $a=1/2$, $bc=2$, $b+c=7/2$, i.e.

$$(a:b:c)=\left(\frac{1}{2}:\frac{7-\sqrt{17}}{4}:\frac{7+\sqrt{17}}{4}\right)$$

and permutations.

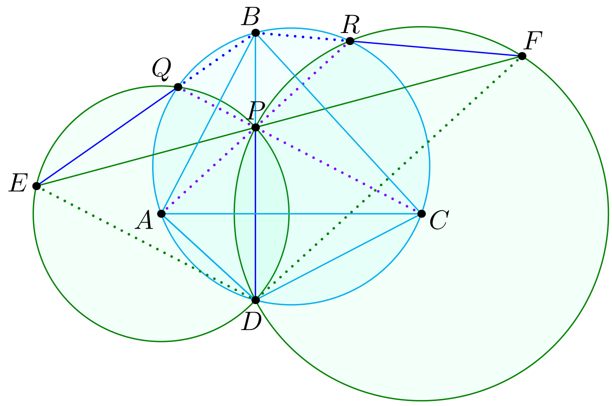
§3 USAJMO 2018/3 (Ray Li)

Problem 3 (USAJMO 2018/3)

Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

Upon viewing the problem with respect to $\triangle ABC$, we actually recognize all the given points; the diagram below should explain everything.

In what follows, we present a full, rigorous solution in as few words as possible.



Line EF is the Steiner line from D in $\triangle ABC$. With $\overline{AC} \perp \overline{BD}$, we have P is the orthocenter of $\triangle ABC$. Redefine Q, R as the reflections of P in $\overline{BA}$, $\overline{BC}$, so that they lie on ω. Then $DPQE$, $DPRF$ are (cyclic) isosceles trapezoids, so Q, R are as described in the problem. Finally $EQ = DP = FR$, the end.

§4 USAJMO 2018/4 (Titu Andreescu)

Problem 4 (USAJMO 2018/4)

Find all real numbers x such that there is a triangle ABC with $\angle ABC \geq 90^\circ$ inscribed in a circle of radius 2, such that if $a = BC$, $b = CA$, $c = AB$, then $x^4 + ax^3 + bx^2 + cx + 1 = 0$.

The answer is $\frac{1}{2}(-\sqrt{6} \pm \sqrt{2})$. The condition $\angle B \geq 90^\circ$ gives $a^2 + c^2 \leq b^2 \leq 4b$, so

$$\begin{aligned} 0 &= x^4 + ax^3 + bx^2 + cx + 1 \\ &= x^2\left(x + \frac{a}{2}\right)^2 + \left(\frac{c}{2}x + 1\right)^2 + \left(b - \frac{a^2 + c^2}{4}\right). \\ &\geq 0 + 0 + 0 = 0, \end{aligned}$$

so equality holds.

In particular, we must have $\frac{a}{2} = -x = \frac{2}{c}$, i.e. $ac = 4$, and $a^2 + c^2 = b^2 = 16$. Solving gives $\{a, c\} = \{\sqrt{6} \pm \sqrt{2}\}$, which work.

§5 USAJMO 2018/5 (Ankan Bhattacharya)

Problem 5 (USAMO 2018/4)
Let p be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k,\ a_2 + 2k,\ \ldots,\ a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

For any two $i < j$, we have

$$a_i + ik \equiv a_j + jk \pmod{p} \iff k \equiv (a_i - a_j)(j - i)^{-1} \pmod{p}.$$

Hence, the number of (i, j, k) with $i < j$ and $a_i + ik \equiv a_j + jk \pmod{p}$ is precisely $\binom{p}{2}$.

By Pigeonhole, for some k there are at most $\frac{p-1}{2}$ pairs $i < j$ with $a_i + ik \equiv a_j + jk \pmod{p}$, thus there are at least $\frac{p+1}{2}$ distinct residues among $a_1 + k$, …, $a_p + pk$.

§6 USAJMO 2018/6 (Maria Monks Gillespie)

Problem 6 (USAJMO 2018/6)

Karl starts with n cards labeled $1, 2, 3, \dots, n$ lined up in a random order on his desk. He calls a pair (a, b) of these cards swapped if $a > b$ and the card labeled a is to the left of the card labeled b.

Karl picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on until he has picked up and put back each of the cards 1, 2, …, n exactly once in that order.

For example, one such process for $n = 4$ is

$$3142 \to 3412 \to 2341 \to 2431 \to 2341.$$

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

Consider another parallel process, such that whenever we move card i, we replace its label with $n + i$. Thus for starting position 3142, the new process is

$$3142 \to 3452 \to 6345 \to 6475 \to 6785.$$

In the final configuration, each card is exactly n more than the final configuration of the original process, so it has the same number of inversions. But the new process preserves the number of inversions at each step, end proof.