IMO 2023

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Last updated July 16, 2023

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§0 Problems

Problem 1. Determine all composite integers $n > 1$ that satisfy the following property: if d_1, d_2, $\ldots$, d_k are all the posiitve divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k-2$.

Problem 2. Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of $\triangle ABC$. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to $\overline{BC}$ meets $\overline{BS}$ at D and meets Ω again at $E \neq A$. The line through D parallel to $\overline{BC}$ meets line BE at L. Denote the circumcircle of $\triangle BDL$ by ω. Let ω meet Ω again at $P \neq B$.

Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Problem 3. For each integer $k \ge 2$, determine all infinite sequences of positive integers a_1, a_2, $\ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where c_0, c_1, $\ldots$, c_{k-1} are nonnegative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

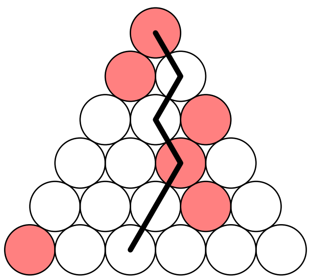
for every integer $n \ge 1$.

Problem 4. Let x_1, x_2, $\ldots$, x_{2023} be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \ge 3034$.

Problem 5. Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly ones of which is colored red. A *ninja path* in a Japanese triangle in a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in the triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Problem 6. Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of triangle ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Lines BC_1 and CB_1 meet at A_2, lines CA_1 and AC_1 meet at B_2, and lines AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2, and CC_1C_2 all pass through three common points.

§1 IMO 2023/1

Problem 1

Determine all composite integers $n > 1$ that satisfy the following property: if d_1, d_2, $\ldots$, d_k are all the posiitve divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k-2$.

The answer is prime powers, which clearly work. Now we show no other n work.

Of course $d_{k-2} \mid d_k = n$, and we are given $d_{k-2} \mid d_{k-1} + d_k$, so $d_{k-2} \mid d_{k-1}$. Since $d_3 d_{k-2} = d_2 d_{k-1} = n$, we also have $d_2 \mid d_3$. Hence if $p = d_2$ is the smallest prime divisor of n, then $d_3 = p^2$.

Let $\ell \ge 4$ be minimal so that d_ℓ is not a power of p. Then $d_i = p^{i-1}$ for $i \le \ell$ and $d_\ell = q$ is another prime (if, for contradiction, n is not a power of p). We have $d_{\ell-2} \mid d_{\ell-1} + d_\ell$, or $p^{\ell-3} \mid p^{\ell-2} + q$, or $p^{\ell-3} \mid q$. Since $\ell - 3 \ge 1$, this is absurd.

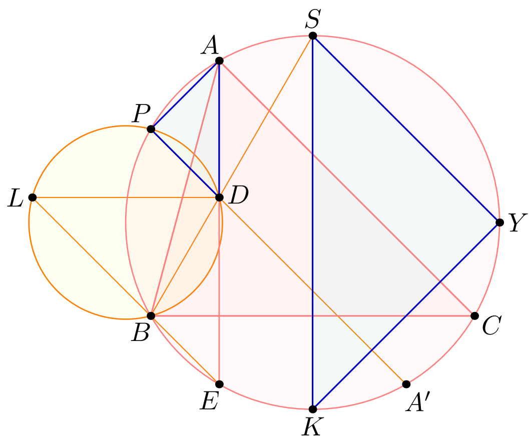
§2 IMO 2023/2

Problem 2

Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of $\triangle ABC$. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to $\overline{BC}$ meets $\overline{BS}$ at D and meets Ω again at $E \neq A$. The line through D parallel to $\overline{BC}$ meets line BE at L. Denote the circumcircle of $\triangle BDL$ by ω. Let ω meet Ω again at $P \neq B$.

Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Let K denote the midpoint of the other arc BC, let A' be the antipode of A, and let the tangent at P to ω intersect Ω at Y. The goal is to show $\overline{DS}$, $\overline{AK}$, $\overline{PY}$ concur.



Note that

- By converse Reim's theorem on Ω and ω, since $\overline{LD} \parallel \overline{EA'}$ we have P, D, A' collinear.
- By Reim's theorem on Ω and ω, we have $\overline{SY} \parallel \overline{PD}$.

But also $\overline{AD} \parallel \overline{SK}$ and $\measuredangle APD = \measuredangle APA' = 90^\circ = \measuredangle KYS$, so $\triangle DAP$ and $\triangle SKY$ are homothetic. This implies the desired concurrence at the center of homothety.

§3 IMO 2023/3 (MAS)

Problem 3

For each integer $k \geq 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_0, c_1, \ldots, c_{k-1}$ are nonnegative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \geq 1$.

The answer is arithmetic progressions with nonnegative common difference d, which satisfy the polynomial $P(x) = (x+d_1)\cdots(x+d_k)$. We now show only arithmetic progressions work.

Claim 1. (a_i) is nondecreasing.

Proof. Assume for contradiction $a_m > a_{m+1}$. Set $n \leftarrow m+1$, and while there exists $n' > n$ with $a_{n'} < a_n$, set $n \leftarrow n'$. This process must terminate, since positive integers cannot decrease forever.

Now a_n has the property that it is less than or equal to every subsequent term of the sequence. While $a_{n-1} \leq a_n$, set $n \leftarrow n-1$, which preserves the aforementioned property. This process must terminate, since $m < n$ and $a_m > a_{m+1}$.

Now we have $a_{n-1} > a_n \leq a_{n+i}$ for all $i \geq 1$. But P is increasing, so $a_{n-1} > a_n$ implies

$$a_n \cdots a_{n+k-1} = P(a_{n-1}) > P(a_n) = a_{n+1}\cdots a_{n+k},$$

or $a_n > a_{n+k}$, contradiction. □

Claim 2. (a_i) is unbounded or constant.

Proof. Suppose (a_i) is bounded and has a maximum a_m. Then

$$a_m^k \leq P(a_m) = a_{m+1}\cdots a_{m+k} \leq a_m^k,$$

so equality holds and thus $P(x) \equiv x^k$ and $a_m = a_{m+1} = \cdots = a_{m+k}$. Repeating with a_{m+k} as our maximum gives that $a_m = a_{m+i}$ for all $i \geq 1$.

Moreover, $a_{m-1}^k = P(a_{m-1}) = a_m \cdots a_{m+k-1} = a_m^k$, so $a_{m-1} = a_m$, and continuing backwards, we also deduce $a_i = a_m$ for all $i < m$. This proves the claim. □

We henceforth assume (a_i) is unbounded.

Claim 3. There is a constant D such that $a_{n+i} - a_n \leq D$ for each n and $i \leq k$.

Proof. For some massive P, we have $P(x) \leq (x+P)^k$ for all $x \geq 1$ (by making each coefficient larger). Then we have

$$(a_n + P)^2 \geq P(a_n) = a_{n+1}\cdots a_{n+k} \geq a_{n+1}^k,$$

so $a_{n+1} - a_n \leq P$. Then take $D = kP$. □

Then there are finitely many possible values of the multiset

$$D_n = \{a_{n+1} - a_n,\ a_{n+2} - a_n,\ \dots,\ a_{n+k} - a_n\},$$

so by infinite Pigeonhole there is an $S = \{d_1, \dots, d_k\}$ such that $D_n = S$ for infinitely many n and thus infinitely many values of a_n; that is,

$$P(a_n) = (a_n + d_1) \cdots (a_n + d_k)$$

for infinitely many values of a_n, implying

$$P(x) = (x + d_1) \cdots (x + d_k)$$

as a polynomial identity.

Moreover, all of the other finitely many possible multisets each occurs finitely often, else by the same argument, we would find $P(x)$ equal to another polynomial, hence for some N, we have $D_n = S$ for $n \ge N$.

Let $d_1 \le \cdots \le d_k$. Since (a_i) is increasing, we have $a_{n+i} = a_n + d_i$ for all $n \ge N$ and $i \le k$. In particular,

$$a_n + d_i = a_{n+i} = \underbrace{(\underbrace{(\underbrace{(a_n + d_1)}_{a_{n+1}} + d_1)}_{a_{n+2}} + \cdots)}_{a_{n+i-1}} + d_1 = a_n + id_1,$$

implying $d_i = id_1$ for all $i \le k$. Let $d = d_1$ for convenience.

Now we have $a_{n+1} - a_n = d$ for all $n \ge N$. Then

$$P(a_{N-1}) = a_N \cdots a_{N+k-1} = a_N(a_N + d) \cdots (a_N + (k-1)d) = P(a_N - d),$$

and since P is increasing on $(-d, \infty)$, we have $a_{N-1} = a_N - d$ (and thus $a_{N-1} > 0$). Continuing downward, we conclude $a_{n+1} - a_n = d$ for all n, and we are done.

§4 IMO 2023/4

Problem 4

Let x_1, x_2, …, x_{2023} be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$.

We prove:

Claim 1. $a_n - a_{n-1} \geq 1$ for all $n \geq 2$, with equality iff

$$x_n^2 = \frac{x_1 + \cdots + x_{n-1}}{\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}}}.$$

Proof. We have

$$\begin{aligned} a_n^2 &= (x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \\ &= a_{n-1}^2 + 1 + \frac{1}{x_n}(x_1 + \cdots + x_{n-1}) + x_n\left(\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}}\right) \\ &\geq a_{n-1}^2 + 1 + 2a_{n-1} = (a_{n-1}+1)^2 \end{aligned}$$

by AM-GM, and the equality case is clear. □

Claim 2. $a_n - a_{n-2} \geq 3$ for all $n \geq 3$.

Proof. We show equality for $a_{n-1} - a_{n-2} \geq 1$ and $a_n - a_{n-1} \geq 1$ cannot both hold. If so, we would have

$$\frac{x_{n-1}}{1/x_{n-1}} = \frac{x_1 + \cdots + x_{n-2}}{\frac{1}{x_1} + \cdots + \frac{1}{x_{n-2}}} = \frac{x_1 + \cdots + x_{n-1}}{\frac{1}{x_1} + \cdots + \frac{1}{x_{n-1}}} = \frac{x_n}{1/x_n},$$

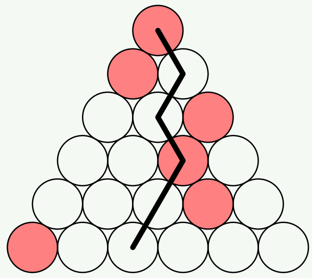
implying $x_{n-1} = x_n$, contradiction. (The second equality above follows since summing numerators and denominators of equal fractions preserves value.) □

Since $a_1 = 1$ it follows that $a_{2k+1} \geq 3k + 1$ for all $k \geq 1$.

§5 IMO 2023/5

Problem 5

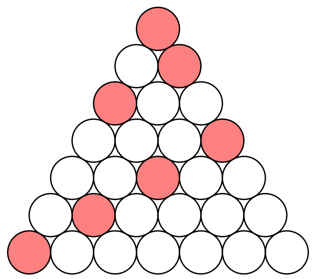
Let n be a positive integer. A *Japanese triangle* consists of $1+2+\cdots+n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly ones of which is colored red. A *ninja path* in a Japanese triangle in a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in the triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

The answer is $\lfloor \log_2 n \rfloor + 1$. We impose a coordinate system in which (i, j) with $i, j \geq 1$ denotes the jth ball in the ith row.

Upper bound Let $k = \lfloor \log_2 n \rfloor + 1$. We color red the cells $(2^j + i, 2^j - i)$ for $0 \leq i < 2^j$ and $2^j + i \leq n$ as shown.



In each of the k intervals of rows $[2^j, 2^{j+1})$ (with the last interval possibly truncated), the second coordinate of the red cells is decreasing, but in any ninja path the second coordinate is nondecreasing, so any ninja path may obtain at most one red cell from each of these k intervals.

First proof of lower bound Let $f(i, j)$ denote the maximum number of red cells on a path from $(0, 0)$ to (i, j). Let

$$S_n = \sum_{j=0}^{i} f(n, j)$$

denote the sum of f across the nth row.

Claim 1. For each n, we have

$$S_{n+1} \geq S_n + 1 + \left\lceil \frac{S_n}{n} \right\rceil .$$

Proof. Let j be so that $f(n,j)$ is maximal, so $f(n,j) \geq \frac{S_n}{n}$. Then

- For $i \leq j$, we have $f(n+1,i) \geq f(n,i)$, by considering the path to $(n+1,i)$ that passes through (n,i) and maximizes the number of red cells.
- For $i > j$, we have $f(n+1,i) \geq f(n,i-1)$ by considering the path to $(n+1,i)$ that passes through $(n,i-1)$ and maximizes the number of red cells.

Moreover, there is a i for which $(n+1,i)$ is red, in which case equality does not hold for the respective bound given above.

Therefore,

$$\begin{aligned} S_{n+1} &\geq 1 + \sum_{1\leq i\leq j} f(n+1,i) + \sum_{j<i\leq n+1} f(n+1,i) \\ &\geq 1 + \sum_{i\leq j} f(n,i) + \sum_{j\leq i\leq n} f(n,i) \\ &= 1 + f(n,j) + S_n \geq 1 + \frac{n+1}{n} \cdot S_n, \end{aligned}$$

as needed. □

Claim 2. For each $n = 2^k + i$ with $0 \leq i < 2^k$, we have

$$S_n \geq nk + 2i + 1.$$

Proof. The proof is induction on n, with base case $S_1 = 1$. We note:

- In general, if $S_n \geq nk + i + 1$, then the claim gives

$$S_{n+1} \geq (nk + 2i + 1) + 1 + (k+1) = (n+1)k + 2(i+1) + 1.$$

- If $n = 2^{k+1} - 1$ and $S_n \geq nk + (2^{k+1} - 1) = n(k+1)$ then

$$S_{n+1} \geq n(k+1) + 1 + (k+1) = (n+1)(k+1) + 1.$$ □

In particular there always exists j with

$$f(n,j) \geq \left\lceil \frac{S_n}{n} \right\rceil \geq k+1,$$

as desired.

Remark. It is also possible to prove $S_{2^k} \geq k \cdot 2^k + 1$ directly by induction on k. This suffices.

Second proof of lower bound (Yotam Amir) Let $g(i,j)$ denote the maximum number of red cells on a path from (i,j) to the nth row. Let

$$T_n = \sum_{j=0}^{i} 2^{f(n,j)}.$$

Claim. $T_i \geq T_{i+1}$ for all k.

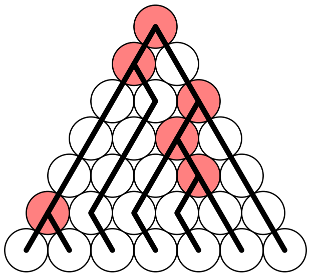
Proof. Let (i, j) be red. Then,

- For all $j' < j$, we have $f(i, j') \geq f(i+1, j')$
- For all $j' > j$, we have $f(i, j') \geq f(i+1, j'+1)$.
- We have $f(i, j) \geq 1 + \max\{f(i+1, j), f(i+1, j+1)\}$ and thus $2^{f(i,j)} \geq 2^{f(i+1,j)} + 2^{f(i+1,j+1)}$.

The claim follows. □

Then $T_0 \geq T_n = n + 1$, so $f(0,0) \geq \log_2(n+1)$, as needed.

Third proof of lower bound (Iran team leader) Add an empty $(n+1)$th row, and draw the unique binary tree with root $(0,0)$ where every red circle has two children (directly below it) and every other circle has one child.



Suppose each path from $(0,0)$ to the $(n+1)$th row in this tree contains at most k red circles.

Consider the map
$$\{0,1\}^k \to \{\text{cells in } (n+1)\text{th row}\}$$
where each $x \in \{0,1\}^k$ is a series of instructions on how to build a path from $(0,0)$ to the $(n+1)$th row, so that at the ith red circle we encounter in the path, we go left if the ith bit is 0 and right if the ith bit is 1. (Once we reach the nth row, any unused bits are ignored.) Each path may be represented in this form, so this is a surjection.

We therefore conclude $2^k \geq n+1$.

§6 IMO 2023/6 (USA)

Problem 6

Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of triangle ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Lines BC_1 and CB_1 meet at A_2, lines CA_1 and AC_1 meet at B_2, and lines AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2, and CC_1C_2 all pass through three common points.

Note that for a point P in the interior of $\triangle ABC$ with $\angle PCB = r$ and $\angle PBC = s$, trig Ceva gives that the barycentric coordinates of P are given by

$$\begin{aligned} P &= \left(1 : \frac{\sin(60^\circ - r)}{\sin r} : \frac{\sin(60^\circ - s)}{\sin s}\right) \\ &= \left(1 : \frac{\sqrt{3}}{2}\cot r - \frac{1}{2} : \frac{\sqrt{3}}{2}\cot s - \frac{1}{2}\right). \end{aligned}$$

A circle through A has equation of the form $xy + yz + zx = (x + y + z)(vy + wz)$, and P lying on this circle gives

$$\begin{aligned} \left(\frac{\sqrt{3}}{2}\cot r - \frac{1}{2}\right)v + \left(\frac{\sqrt{3}}{2}\cot s - \frac{1}{2}\right)w &= \frac{\left(\frac{\sqrt{3}}{2}\cot r + \frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\cot s + \frac{1}{2}\right) - 1}{\frac{\sqrt{3}}{2}(\cot r + \cot s)} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\cot r \cot s - 1}{\cot r + \cot s} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\cot(r + s). \end{aligned}$$

Let $\alpha = \angle A_1BC$ and define β and γ similarly, so we are given $\alpha + \beta + \gamma = 30^\circ$. Now we have

$$\begin{aligned} A_1 &= \left(1 : \frac{\sqrt{3}}{2}\cot\alpha - \frac{1}{2} : \frac{\sqrt{3}}{2}\cot\alpha - \frac{1}{2}\right) \\ \text{and} \quad A_2 &= \left(1 : \frac{\sqrt{3}}{2}\cot(60^\circ - \beta) - \frac{1}{2} : \frac{\sqrt{3}}{2}\cot(60^\circ - \gamma) - \frac{1}{2}\right) \end{aligned}$$

If we write the equation of (AA_1A_2) as $xy + yz + zx = (x + y + z)(vy + wz)$, the condition $A_1 \in (AA_1A_2)$,

$$\begin{aligned} \left(\frac{\sqrt{3}}{2}\cot\alpha - \frac{1}{2}\right)(v + w) &= \frac{1}{2} + \frac{\sqrt{3}}{2}\cot(2\alpha) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\cot^2\alpha - 1}{2\cot\alpha} \\ &= \frac{\left(\frac{\sqrt{3}}{2}\cot\alpha - \frac{1}{2}\right)\left(\frac{1}{2}\cot\alpha + \frac{\sqrt{3}}{2}\right)}{\cot\alpha} \\ \implies v + w &= \frac{1}{2} + \frac{\sqrt{3}}{2}\tan\alpha = \frac{\cos(60^\circ - \alpha)}{\cos\alpha}. \end{aligned}$$

The condition $A_2 \in (AA_1A_2)$ gives

$$\begin{aligned}
\left(\frac{\sqrt{3}}{2}\cot(60^\circ-\beta)-\frac{1}{2}\right)v+\left(\frac{\sqrt{3}}{2}\cot(60^\circ-\gamma)-\frac{1}{2}\right)w &= \frac{1}{2}+\frac{\sqrt{3}}{2}\cot(120^\circ-\beta-\gamma)\\
&= \frac{1}{2}+\frac{\sqrt{3}}{2}\cot(90^\circ+\alpha)\\
&= \frac{1}{2}-\frac{\sqrt{3}}{2}\tan\alpha\\
\implies \frac{\sqrt{3}}{2}\cot(60^\circ-\beta)v+\frac{\sqrt{3}}{2}\cot(60^\circ-\gamma)w &= \left(\frac{1}{2}-\frac{\sqrt{3}}{2}\tan\alpha\right)+\frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\tan\alpha\right)\\
&= \frac{3}{4}-\frac{\sqrt{3}}{4}\tan\alpha\\
\implies \cot(60^\circ-\beta)v+\cot(60^\circ-\gamma)w &= \frac{\sqrt{3}}{2}-\frac{1}{2}\tan\alpha=\frac{\cos(30^\circ+\alpha)}{\cos\alpha}.
\end{aligned}$$

Solving, we have

$$\begin{aligned}
v &= \frac{\frac{\cos(30^\circ+\alpha)}{\cos\alpha}-\cot(60^\circ-\gamma)\frac{\cos(60^\circ-\alpha)}{\cos\alpha}}{\cot(60^\circ-\beta)-\cot(60^\circ-\gamma)}\\
&= \frac{\sin(60^\circ-\beta)(\cos(30^\circ+\alpha)\sin(60^\circ-\gamma)-\cos(60^\circ-\gamma)\cos(60^\circ-\alpha))}{\cos\alpha(\cos(60^\circ-\beta)\sin(60^\circ-\gamma)-\cos(60^\circ-\gamma)\sin(60^\circ-\beta))}\\
&= \frac{-\sin(60^\circ-\beta)\cos(120^\circ-\alpha-\gamma)}{\cos\alpha\sin(\beta-\gamma)}=\frac{\sin(60^\circ-\beta)\sin\beta}{\cos\alpha\sin(\beta-\gamma)}
\end{aligned}$$

and thus

$$vy+wz=\frac{\sin(60^\circ-\beta)\sin\beta}{\cos\alpha\sin(\beta-\gamma)}y-\frac{\sin(60^\circ-\gamma)\sin\gamma}{\cos\alpha\sin(\beta-\gamma)}z.$$

Now write the equation of (AA_1A_2) as $0=-(xy+yz+zx)+(x+y+z)(u_ax+v_ay+w_az)$, and similarly express (BB_1B_2) and (CC_1C_2). (Of course $u_a=v_b=w_c=0$.) To show these three circles are coaxial, it will suffice to show the existence of λ_1, λ_2, λ_3 such that

$$\lambda_1\cdot(AA_1A_2)+\lambda_2\cdot(BB_1B_2)+\lambda_3\cdot(CC_1C_2)=0,$$

or rather so that

$$\begin{aligned}
\lambda_1+\lambda_2+\lambda_3&=0\\
\lambda_1u_a+\lambda_2u_b+\lambda_3u_c&=0\\
\lambda_1v_a+\lambda_2v_b+\lambda_3v_c&=0\\
\lambda_1w_a+\lambda_2w_b+\lambda_3w_c&=0.
\end{aligned}$$

It is clear that the choice of $\lambda_1=\cos\alpha\sin(\beta-\gamma)$, and λ_2 and λ_3 symetrically satisfies the last three equations, and we also have

$$\lambda_1=\cos\alpha\sin\beta\cos\gamma-\cos\alpha\cos\beta\sin\gamma$$

which cyclically sums to zero.

Now it is not hard to check that the three circles intersect, so we are done.