IMO 2021

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§0 Problems

Problem 1. Let $n \geq 100$ be an integer. Ivan writes the numbers n, $n+1$, ..., $2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Problem 2. Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers x_1, ..., x_n.

Problem 3. Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. Point E on segment AC satisfies $\angle ADE = \angle BCD$, point F on segment AB satisfies $\angle FDA = \angle DBC$, and point X on line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Problem 4. Let Γ be a circle with center I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD, and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Problem 5. Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favorite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the kth move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the kth move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Problem 6. Let $m \geq 2$ be an integer, A be a finite set of (not necessarily positive) integers and B_1, B_2, B_3, ..., B_m be subsets of A. Assume that for every $k = 1, 2, \ldots, m$ the sum of the elements of B_k is m^k. Prove that A contains at least $m/2$ elements.

§1 IMO 2021/1

Problem 1

Let $n \geq 100$ be an integer. Ivan writes the numbers n, $n+1$, ..., $2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

It will suffice to find cards a, b, c such that $a+b$, $b+c$, $c+a$ are perfect squares. Indeed the cards

$$2k^2 - 4k < 2k^2 + 1 < 2k^2 + 4k$$

should work, so long as the three numbers are in the range $[n, 2n]$.

We will show we may select k such that they are in the range, whenever $n \geq 100$. Let $k = \lfloor \sqrt{n} \rfloor - 1$, so that $2k^2 + 4k < 2(k+1)^2 \leq 2n$, and moreover since $n \leq k^2 + 4k + 3$ we have

$$n \leq k^2 + 4k + 3 \leq 2k^2 - 4k$$

since $k \geq 9$.

§2 IMO 2021/2

Problem 2

Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i-x_j|}\le\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i+x_j|}$$

holds for all real numbers $x_1, \ldots, x_n$.

The idea is to force $x_i+x_j=0$ for some i, j, then induct down.

Claim. If $x_i+x_j\neq 0$ for all i, j, then for some k the sequence $x_1+k, \ldots, x_n+k$ produces the same left-hand side but a smaller right-hand side.

Proof. Evidently shifting the x_i by a constant does not affect the left-hand side.

Take small ε, and observe by Jensen that

$$\left(\sum_{i,j}\sqrt{|x_i+x_j+2\varepsilon|}+\sqrt{|x_i+x_j-2\varepsilon|}\right)\le\sum_{i,j}2\sqrt{|x_i+x_j|},$$

so either

$$\sum_{i,j}\sqrt{|x_i+x_j+2\varepsilon|}\le\sum_{i,j}\sqrt{|x_i+x_j|}\quad\text{or}\quad\sum_{i,j}\sqrt{|x_i+x_j-2\varepsilon|}\le\sum_{i,j}\sqrt{|x_i+x_j|}.$$

Thus either $k=\varepsilon$ or $k=-\varepsilon$ works. □

Therefore we reduce the problem to the cases where $x_i+x_j=0$ for some i, j. Finally, deleting $x_i=+x$ and $x_j=-x$ decreases both sides by

$$2\sqrt{2x}+\sum_{k\neq i,j}\left(2\sqrt{|x_k-x|}+2\sqrt{|x_k+x|}\right),$$

so we may induct down. The remaining base cases of $n=0,1$ are trivial.

Remark. The above proof shows that the equality cases of this inequality are when the x_i are symmetric across 0.

§3 IMO 2021/3

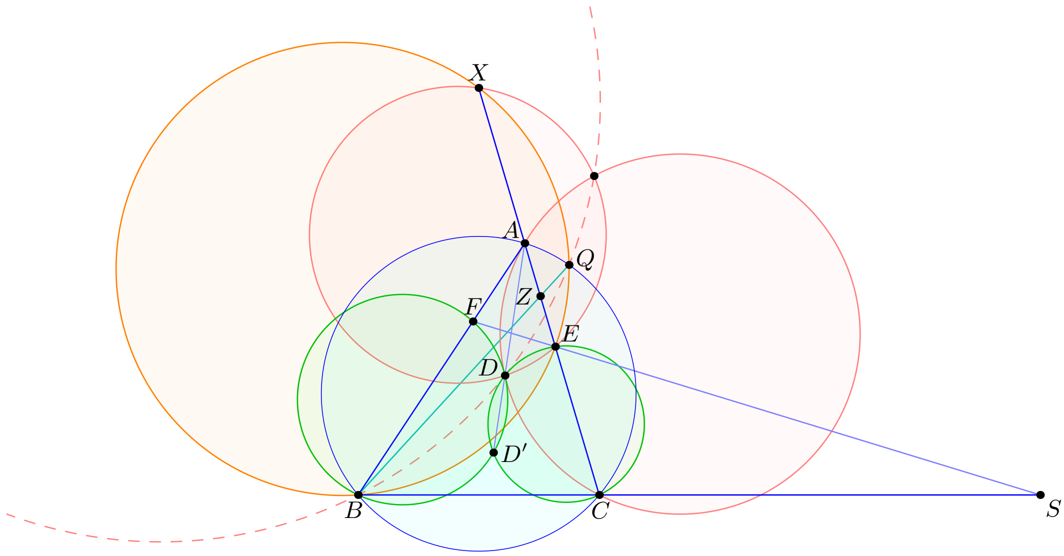
Problem 3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. Point E on segment AC satisfies $\angle ADE = \angle BCD$, point F on segment AB satisfies $\angle FDA = \angle DBC$, and point X on line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Let D' be the isogonal conjugate of D.

Claim 1. $BD'DF$, $CD'DE$, and $BCEF$ are cyclic.

Proof. The angle conditions give $BD'DF$ and $CD'DE$ cyclic. It follows that $AB \cdot AF = AD \cdot AD' = AC \cdot AE$, so quadrilateral $BCEF$ is cyclic. □



Let $S = \overline{BC} \cap \overline{EF}$.

Claim 2. (DBC) and (DEF) are tangent at D to line SD.

Proof. Observe that

$$\begin{aligned}\measuredangle BDF &= \measuredangle BDA + \measuredangle D'DF \\ &= \measuredangle DBA + \measuredangle BAD + \measuredangle D'BF \\ &= \measuredangle CBD' + \measuredangle BAD + \measuredangle D'BA \\ &= \measuredangle CBA + \measuredangle BAD \\ &= \measuredangle(\overline{AD}, \overline{EF}) \\ &= \measuredangle D'DE + \measuredangle DEF \\ &= \measuredangle BCD + \measuredangle DEF,\end{aligned}$$

so (DBC) and (DEF) are tangent at D. The common tangent passes through S by radical axis theorem with $(BCEF)$. □

Let Q be the Miquel point of $BCEF$, so inversion Ψ at S with radius SD will swap (B, C), (E, F), (A, Q).

Claim 3. $BEQX$ is cyclic.

Proof. This is just an angle chase:

$$\measuredangle XBQ = \measuredangle XBC + \measuredangle CBQ = \measuredangle BCA + \measuredangle CAQ = \measuredangle CSQ = \measuredangle XEQ. \qquad \square$$

Claim 4. (DEX), (ADC), (QDB) are coaxial.

Proof. We will show $Z = \overline{AC} \cap \overline{BQ}$ lies on the common radical axis. But

$$\begin{aligned} \operatorname{Pow}(Z, (ADC)) &= ZA \cdot ZC = ZB \cdot ZQ = \operatorname{Pow}(Z, (QDB)) \\ \operatorname{Pow}(Z, (DEX)) &= ZE \cdot ZX = ZB \cdot ZQ = \operatorname{Pow}(Z, (QDB)), \end{aligned}$$

as desired. $\square$

Let line ℓ contain the centers of (DEX), (ADC), (QDB). Note that Ψ swaps (ADC) and (QDB), so S lies on ℓ. This gives the desired conclusion.

§4 IMO 2021/4

Problem 4

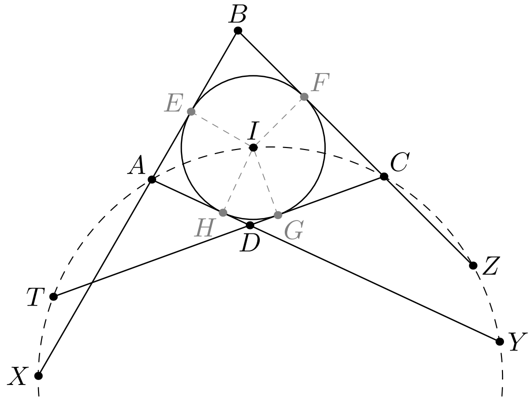
Let Γ be a circle with center I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD, and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Note since $\overline{AI}$ bisects $\angle XAY$ that $IX = IY$, and analogously $IT = IZ$. It follows that $TX = YZ$.

Now if E, F, G, H are the feet from I to $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, note the congruences $\triangle IEX \cong \triangle IHY$ and $\triangle IGT \cong \triangle IFZ$, from which we obtain

$$\begin{aligned}(AD - CD) + (XA - CZ) + (DT - DY) &= (AB - CB) + (XA - CZ) + (DT - DY) \\ &= (BX - BZ) + (DT - DY) \\ &= (EX - FZ) + (GY - HY) = 0.\end{aligned}$$



§5 IMO 2021/5

Problem 5

Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favorite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the kth move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the kth move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Call a walnut k *conservative* if on the kth move Jumpy swaps walnuts a and b with $a, b < k$, and call a walnut k *liberal* if on the kth move Jumpy swaps walnuts a and b with $a, b > k$. Assume for contradiction every walnut is either conservative or liberal.

Claim 1. If walnut k is conservative, then it does not move after the kth move.

Proof. Indeed, I contend the neighbors of k will always be smaller than the move number. After the kth move, the neighbors of k are $a, b < k$ by design. If on move t, a neighbor of k is swapped out, since the original neighbor was less than t, the new neighbor is also less than t. $\square$

Claim 2. If k is conservative, then its neighbors in the final position are not conservative.

Proof. If the kth move swaps a and b, with $a, b < k$, then a and b are not conservative since they move on the kth move. Whenever either neighbor of k is replaced, each resulting neighbor i is not conservative, since it is swapped after the ith move. $\square$

It follows that there are at most 1010 conservative walnuts. By reversing the process, we see analogously that there are at most 1010 liberal walnuts, contradicting that all walnuts are either conservative or liberal.

§6 IMO 2021/6

Problem 6

Let $m \geq 2$ be an integer, A be a finite set of (not necessarily positive) integers and B_1, B_2, B_3, ..., B_m be subsets of A. Assume that for every $k = 1, 2, \ldots, m$ the sum of the elements of B_k is m^k. Prove that A contains at least $m/2$ elements.

If $A = \{a_1, \ldots, a_n\}$ has n elements then we are given each m^k may be expressed in the form

$$m^k = \varepsilon_{k1}a_1 + \varepsilon_{k2}a_2 + \cdots + \varepsilon_{kn}a_n \quad \text{where} \quad 0 \leq \varepsilon_i \leq 1.$$

Consider the m^m nonnegative multiples of m less than m^{m+1}. Each $mj < m^{m+1}$ may be represented in base m as

$$mj = d_m \cdot m^m + d_{m-1} \cdot m^{m-1} + \cdots + d_1 \cdot m \quad \text{where} \quad 0 \leq d_i \leq m-1.$$

Now consider

$$w_j := \sum_{i=1}^{m} d_i \varepsilon_{ij} \leq m(m-1),$$

defined so that

$$mj = w_1a_1 + w_2a_2 + \cdots + w_na_n \quad \text{where} \quad 0 \leq w_i \leq m(m-1).$$

Expressions of the above form may describe at most $(m^2 - m + 1)^n$ distinct numbers, so we have the inequality

$$m^m \leq (m^2 - m + 1)^n < m^{2n}.$$

from which it is clear $n \geq m/2$.

Remark. It turns out we can show $|A| \geq (\frac{2}{3} - o(1))m$; see `https://artofproblemsolving.com/community/c6h2625864p22704660`.