IMO 2020

Compiled by Eric Shen

Last updated June 19, 2021

Contents

§0 Problems

Problem 1. Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove that the following three lines meet in a point: the internal bisectors of $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Problem 2. The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a+b+c+d = 1$. Prove that

$$(a + 2b + 3c + 4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

Problem 3. There are $4n$ pebbles of weights 1, 2, 3, ..., $4n$. Each pebble is colored in one of n colors and there are four pebbles of each color. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

- The total weights of both piles are the same.
- Each pile contains two pebbles of each color.

Problem 4. There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable care companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no immediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are *linked* by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed).

Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem 5. A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers of each pair of cards is also the geometric mean of the numbers on some collection of one or more cards.

For which n does it follow that the numbers on the cards are all equal?

Problem 6. Prove that there exists a positive constant c such that the following statement is true: Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. It follows that there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

(A line ℓ separates a set of points S if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

§1 IMO 2020/1 (POL)

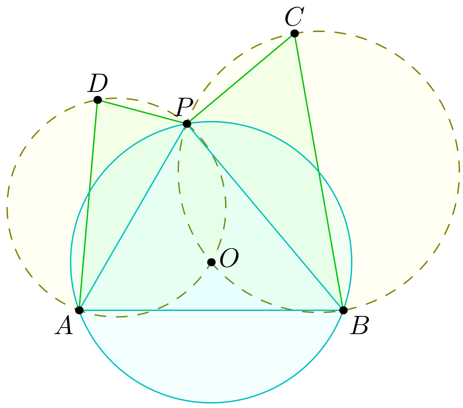
Problem 1

Consider the convex quadrilateral $ABCD$. The point P is in the interior of $ABCD$. The following ratio equalities hold:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Prove that the following three lines meet in a point: the internal bisectors of $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of segment AB.

Let O be the circumcenter of $\triangle PAB$. We will show the three lines meet at O.



Evidently O lies on the perpendicular bisector of $\overline{AB}$.

Note that $\angle AOP = 2\angle ABP = 4\alpha$ and $\angle ADP = 180^\circ - 4\alpha$, so O lies on the circumcircle of $\triangle ADP$. Since $OP = OA$ and O lies on the opposite side of $\overline{AP}$ to D, it follows that O lies on the internal angle bisector of $\angle ADP$.

Similarly O lies on the internal angle bisector of $\angle BCP$, so the desired concurrence point is O.

§2 IMO 2020/2 (BEL)

Problem 2

The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a+b+c+d=1$. Prove that

$$(a+2b+3c+4d)\, a^a\, b^b\, c^c\, d^d < 1.$$

The $a^a b^b c^c d^d$ term is highly irrelevant: just observe that

$$\begin{aligned}
(a+2b+3c+4d)\, a^a\, b^b\, c^c\, d^d &< (a+2b+3c+4d)\left(a^2+b^2+c^2+d^2\right) \\
&\leq (a+3b+3c+3d)\left(a^2+b^2+c^2+d^2\right) \\
&= (a+b+c+d)\left(a^2+b^2+c^2+d^2\right) \\
&\qquad + 2\left(a^2+b^2+c^2+d^2\right)(b+c+d) \\
&\leq (a+b+c+d)\left(a^2+b^2+c^2+d^2\right) \\
&\qquad + 2a(a+b+c+d)(b+c+d) \\
&= (a+b+c+d)\left(a^2+b^2+c^2+d^2+2a(b+c+d)\right) \\
&< (a+b+c+d)^3 = 1.
\end{aligned}$$

Remark. It is not hard to directly verify

$$(a+2b+3c+4d)\left(a^2+b^2+c^2+d^2\right) < 1$$

via expansion.

§3 IMO 2020/3 (USA)

Problem 3

There are $4n$ pebbles of weights 1, 2, 3, ..., $4n$. Each pebble is colored in one of n colors and there are four pebbles of each color. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

- The total weights of both piles are the same.
- Each pile contains two pebbles of each color.

We will show it is possible to find a subset S of the pebbles such that

- S contains two pebbles of each color, and
- for each $i \le 2n$, the pebble of weight i is in S if and only if $(4n+1) - i$ is in S.

Consider a (not necessarily simple) graph G on n vertices, each representing one of the n colors. For each $i \le 2n$, if the pebble of weight i has color a and the pebble of weight $(4n+1) - i$ has color b, draw an edge from a to b. Thus, each $i \le 2n$ represents an edge of G.

Then G is a 4-regular graph, so by taking alternating edges of Eulerian circuits of each connected component, there is a 2-regular subgraph G' with the same vertex set as G. If we let S contain each edge that is present in G', then S obeys the desired conditions. End proof.

§4 IMO 2020/4 (IND)

Problem 4

There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable care companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no immediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are *linked* by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed).

Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

The smallest k is $n^2 - n + 1$.

Proof $k = n^2 - n$ fails: Label the stations 1, 2, ..., n^2 from bottom-up. For company A, let the cables link stations i to $i + 1$ for all i not divisible by n. For company B, let the cables link stations i to $i + n$ for all i. Thus, each company operates k cable cars, yet no two stations are linked by both companies.

Proof $k = n^2 - n + 1$ works: Focus on company A for now, and let the cable cars be edges on a graph with n^2 vertices. Let there be a connected components, each of which is in the shape of a "chain." Since the graph has $n^2 - n + 1$ edges, there are at least $n - 1$ connected components. Similarly let b be the number of chains among B's cable cars, so $a \le n - 1$ and $b \le n - 1$.

Since there are $a \le n - 1$ chains among A's cable cars, at least one such chain ℓ contains at least $n + 2$ vertices. By Pigeonhole, at least one chain of B's cable cars contains two vertices of ℓ, so those two vertices are linked by both A and B.

§5 IMO 2020/5 (EST)

Problem 5

A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers of each pair of cards is also the geometric mean of the numbers on some collection of one or more cards.

For which n does it follow that the numbers on the cards are all equal?

All n. Let the numbers on the cards be $a_1 \geq a_2 \geq \cdots \geq a_n$, and assume by scaling that a_1, $\dots$, a_n are relatively prime.

Claim. Let p be prime. If $p \mid a_1$, then $p \mid a_k$ for all k.

Proof. We will prove $p \mid a_k$ by strong induction on k; assume that p divides a_1 through a_k. We know from the problem condition that

$$\frac{a_k + a_{k+1}}{2} = \sqrt[t]{a_{i_1} a_{i_2} \cdots a_{i_t}}$$

for $i_1 < i_2 < \cdots < i_t$.

If $a_k = a_{k+1}$, then $p \mid a_{k+1}$ as well. Otherwise $a_k < a_{k+1}$, so $i_1 \leq k$. Recall that $p \mid a_{i_1}$, so p divides a_{k+1} as well. $\square$

Therefore no primes divide a_1, so $a_1 = a_2 = \cdots = a_n = 1$.

§6 IMO 2020/6 (TWN)

Problem 6

Prove that there exists a positive constant c such that the following statement is true: Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. It follows that there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $cn^{-1/3}$.

(A line ℓ separates a set of points S if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

Let D be the maximum distance between two points in $\mathcal{S}$. Let these two points be A and B, and let ℓ be the line AB. If we consider the projections of the points in $\mathcal{S}$ onto ℓ, let δ be the maximum distance between two adjacent projections. I contend $\delta \geq O(n^{-1/3})$. It will then suffice to take the perpendicular bisector of these two projections.

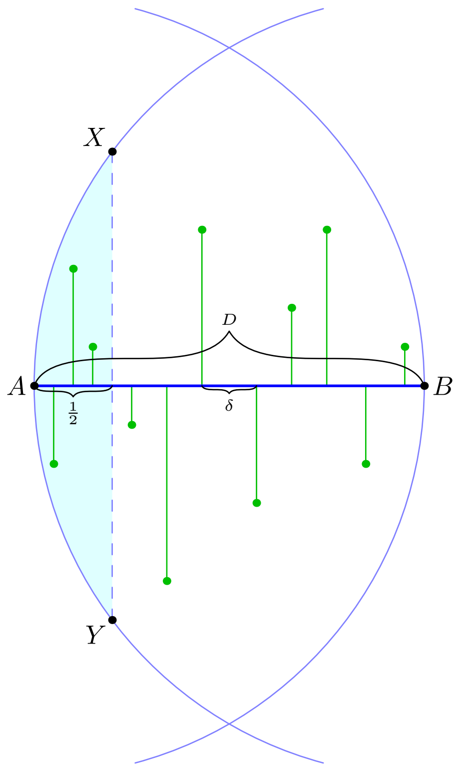
Case 1: $D \geq O(n^{2/3})$. There are n projections on a segment of length $D \geq O(n^{2/3})$, so some two adjacent projections are

$$\delta \geq \frac{D}{n} \geq \frac{O(n^{2/3})}{n} = O(n^{-1/3})$$

apart, as desired.

Case 2: $D \leq O(n^{2/3})$. Consider the disks Γ_A and Γ_B centered at A and B with radii D. Then all the points in $\mathcal{S}$ lie in $\Gamma_A \cap \Gamma_B$.

Consider a chord XY of Γ_B a distance of $1/2$ away from A, and consider the segment $\mathcal{R}$ of Γ_B bounded by $\overline{XY}$ and containing A.



Claim. $\mathcal{R}$ contains at most $O(\sqrt{D})$ points.

Proof. The Pythagorean theorem gives $XY = O(\sqrt{D})$. Since points in $\mathcal{R}$ are at least 1 apart and the width of $\mathcal{R}$ is $\frac{1}{2}$, their projections onto XY must be at least $\frac{\sqrt{3}}{2}$ apart, implying there are at most $O(\sqrt{D})$ points in $\mathcal{R}$. $\square$

The projections of the points in $\mathcal{R}$ onto ℓ cover a segment of length at most $\frac{1}{2}$, so some two adjacent projections are

$$\delta \geq \frac{1/2}{\sqrt{D}} \geq \frac{1}{O(n^{2/3})} = O(n^{-1/3})$$

apart as desired.

> **Remark.** The constant 1/3 is optimal, as constructions can be found attaining $O(n^{-1/3} \log n)$.

> **Remark.** Many contestants were able to prove a bound of $O(n^{-1/2})$. This approaches shows $D \geq O(\sqrt{n})$ by drawing disjoint disks of radii $\frac{1}{2}$ around each point and bounding the area of the disk of radius $D + \frac{1}{2}$ containing all the small disks.