IMO 2019

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§0 Problems

Problem 1. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Problem 2. In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $\overline{PQ}$ is parallel to $\overline{AB}$. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Problem 3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

Problem 4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}).$$

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

Problem 6. Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of $\triangle ABC$ is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to $\overline{EF}$ meets ω at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

§1 IMO 2019/1 (SAF)

Problem 1

Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

The answer is $f \equiv 0$ and $f(x) = 2x + q$ for all q. Let $P(a, b)$ denote the assertion. Notice that

- $P(1, x)$ yields $f(2) + 2f(x) = f(f(x+1))$ for all x.
- $P(0, x+1)$ yields $f(0) + 2f(x+1) = f(f(x+1))$ for all x.

Combining these, $f(2) + 2f(x) = f(0) + 2f(x+1)$. Hence, $f(0) \equiv f(2) \pmod 2$ and for all x,

$$f(x+1) - f(x) = \frac{f(2) - f(0)}{2}.$$

The right hand side is constant, so f is linear. Now, set $f(x) = px+q$ and $c = a+b$. Substituting,

$$\begin{aligned} 2pa + q + 2pb + 2q &= p(p(a+b) + q) + q \\ \iff 2p(a+b) + 3q &= p^2(a+b) + pq + q \\ \iff 2q &= p(c(p-2) + q). \end{aligned}$$

Let $c = 0$; then, either $q = 0$ or $p = 2$.

- If $q = 0$, then $0 = cp(p-2)$ for all c, implying $p \in \{0, 2\}$.
- If $p = 2$, then $2q = 2q$, so all such functions work.

It is easy to check that $f \equiv 0$ and $f(x) = 2x + q$ work, so we are done.

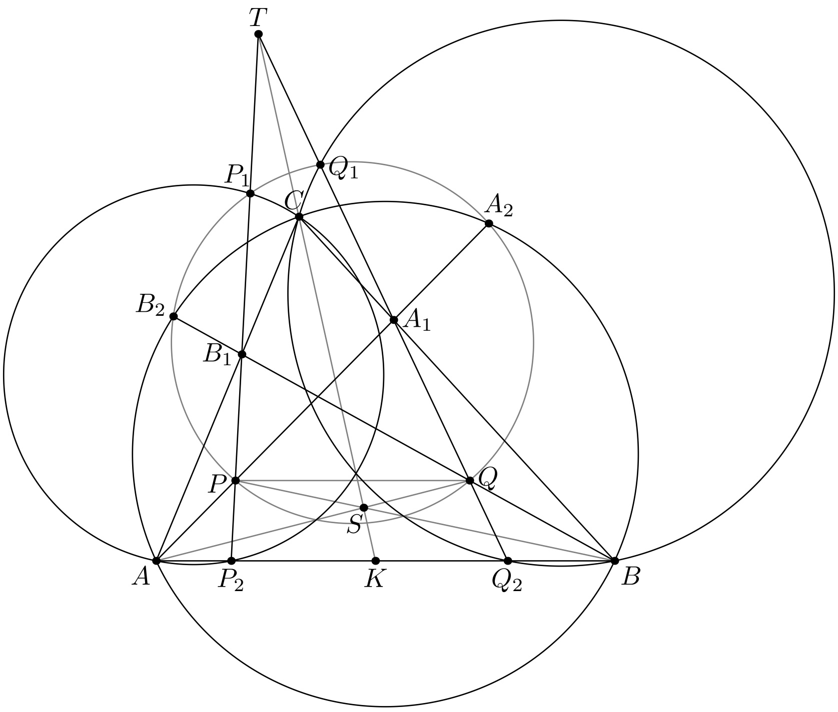
§2 IMO 2019/2 (UKR)

Problem 2

In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that $\overline{PQ}$ is parallel to $\overline{AB}$. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A_1 lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

We present two ways to solve this problem, but both are relatively simple. Thus we provide a single diagram that encompasses both solutions.



First solution, by Reim's Theorem Let $\overline{PP_1}$ and $\overline{QQ_1}$ intersect $\overline{AB}$ at P_2 and Q_2 respectively, and let $\overline{AA_1}$ and $\overline{BB_1}$ intersect the circumcircle of $\triangle ABC$ again at A_2 and B_2 respectively. By the problem statement it is obvious that AP_1CP_2 and BQ_1CQ_2 are cyclic. We claim that P, Q, A_2, B_2, P_1, Q_1 are all concyclic.

Since $\overline{PQ} \parallel \overline{AB}$, by Reim's Theorem PQA_2B_2 is cyclic. Now $B_1P_1 \cdot B_1P_2 = AB_1 \cdot CB_1 = BB_1 \cdot B_1B_2$, whence PB_2P_2B is cyclic. However by Reim's Theorem P_1B_2PQ is also cyclic, so B_2 lies on (PQA_2B_2). Similarly A_2 lies on (PQA_2B_2), so our claim has been proven. Thus P, Q, P_1, Q_1 all lie on a circle, so we are done.

Second solution, by DDIT and Pappus' Theorem Let $P_2 = \overline{AB} \cap \overline{PP_1}$, $Q_2 = \overline{AB} \cap \overline{QQ_1}$, $T = \overline{PP_1} \cap \overline{QQ_1}$, $K = \overline{CT} \cap \overline{AB}$, and $S = \overline{AQ} \cap \overline{BP}$. By construction AP_1CP_2 and BQ_1CQ_2 are cyclic, and by Pappus' Theorem on $\overline{APA_1}$ and $\overline{BQB_1}$, points S, C, T are collinear.

Applying DDIT on $ABPQ$ from T, an involution swaps $(\overline{TA}, \overline{TP})$, $(\overline{TB}, \overline{TQ})$, $(\overline{T\infty_{AB}}, \overline{TS})$, where ∞_{AB} is the point at infinity on $\overline{AB}$. Projecting onto $\overline{AB}$, an inversion at K swaps (A, P_2) and (B, Q_2). It follows that $KA \cdot KP_2 = KB \cdot KQ_2$, so $\overline{TCSK}$ is the radical axis of (AP_1CP_2) and (BQ_1CQ_2). Now $TP_1 \cdot TP_2 = TQ_1 \cdot TQ_2$, so $P_1P_2Q_2Q_1$ is cyclic.

Finally, by Reim's Theorem, PP_1Q_1Q is cyclic, and we are done.

§3 IMO 2019/3 (HRV)

Problem 3

A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B, user B is also friends with user A. Events of the following kind may happen repeatedly, one at a time:

> Three users A, B, and C such that A is friends with both B and C, but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B, and no longer friends with C. All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

This problem can be expressed in terms of a graph G of size 2019. Call the event described in the problem a *move* on ABC, and call a connected component *exhausted* if it is impossible to perform a move without disconnecting it.

Lemma

If a connected graph G is exhausted, then it is either a tree, a cycle, or a complete graph.

Proof. Assume that G is not a tree, a cycle, or a complete graph. We will show that there exists a legal move. Since G is not a tree, take the smallest cycle C. Then there are two cases to consider.

First assume that C is not a triangle, and hence G contains no triangles. Take an edge $\overline{ab}$, with $a \in C$ and $b \notin C$. Now let $c \in C$ be a neighbor of a. Hence we may move abc, as desired.

Now assume that C is a triangle, and let K be the maximal clique; by hypothesis $K \neq G$. Then since G is connected, we can choose an edge $\overline{ab}$ with $a \in K$ and $b \notin K$. Some vertex c of K is not a neighbor of b, as otherwise b would be connected to every node in K and thus part of K. Thus we may move abc. $\square$

Claim 1. G is connected.

Proof. Assume for the sake of contradiction G has at least 2 connected components. By the Pigeonhole Principle some connected component C has size not exceeding 1009. But every node has either 1009 or 1010 neighbors, a contradiction. $\square$

Claim 2. We can turn G into a tree.

Proof. Obviously G is not a tree, cycle, or complete graph, so by the lemma we can apply moves until G transforms into an exhausted graph G'. But each move preserves the parity of the degree of each node, and since some nodes have odd degree, G' cannot be a cycle. Furthermore the number of edges in G is strictly decreasing as we perform moves, and initially G is not connected, so G' cannot be a complete graph. Hence by the lemma, G' is a tree. $\square$

Now we can easily finish once G is a tree. In particular G is acyclic, and so it will forever be acyclic. Perform arbitrary moves breaking G into smaller subtrees until each subtree has size at most 2. Thus each node has degree at most 1, the end.

§4 IMO 2019/4 (SLV)

Problem 4

Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}).$$

The answer is $(1, 1)$ and $(3, 2)$. These clearly work, so we show that these are the only solutions. First notice that

$$\prod_{i=0}^{n-1} \left(2^n - 2^i\right) = \prod_{i=0}^{n-1} 2^i \left(2^{n-i} - 1\right) = 2^{n(n-1)/2} \prod_{i=1}^{n} \left(2^i - 1\right),$$

thus

$$k = \sum_{i=1}^{\infty} \frac{k}{2^i} > \sum_{i=1}^{\infty} \left\lfloor \frac{k}{2^i} \right\rfloor = \nu_2(k!) = \frac{n(n-1)}{2}.$$

Note that for all odd i, $3 \nmid 2^i - 1$, and furthermore by the Lifting the Exponents Lemma,

$$\nu_3(k!) = \sum_{i=1}^{\lfloor n/2 \rfloor} \nu_3\left(4^i - 1\right) = \sum_{i=1}^{\lfloor n/2 \rfloor} \Big(1 + \nu_3(i)\Big) = \lfloor n/2 \rfloor + \nu_3\left(\lfloor n/3 \rfloor !\right).$$

But clearly $\nu_3\big((3\lfloor n/2 \rfloor + \lfloor n/2 \rfloor)!\big) \geq \lfloor n/2 \rfloor + \nu_3(\lfloor n/2 \rfloor !)$, so

$$\nu_3(k!) \leq \nu_3\big((\lfloor n/2 \rfloor + 3\lfloor n/2 \rfloor)!\big) \leq \nu_3\big((2n)!\big),$$

whence $2n + 2 \geq k > \frac{1}{2}n(n-1)$, or rather $0 > n^2 - 5n - 4$ and thus $n < \frac{1}{2}(5 + \sqrt{41}) < 6$. Then we only need to check $n \leq 5$, which is an easy finite case check that yields the desired answer.

§5 IMO 2019/5 (USA)

Problem 5

The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

The answer is $\frac{1}{4}n(n+1)$. Denote by a_n the answer. Consider a graph G_n of degree 2^n, where each node represents one of the possible coin arrangements. For each node, draw a directed edge to the state that must immediately follow.

We will show that (i) this graph is a tree, and (ii) $a_n = \frac{1}{4}n(n+1)$; this obviously solves the problem. To this end, we use induction. Remark that the base case, $n = 1$, is trivial; in fact the even easier case of $n = 0$ is a valid base case as well. G_n is clearly unique, so we construct G_n in terms of G_{n-1}.

Say that the root of G_n is TTT...TT, and that for any node M, $\overline{M}$ denotes the node that results from flipping over every coin, M^r the node the results from reversing the ordering of the coins, and M_1M_2 the node that results from concatenating nodes M_1 and M_2. Also define these operators on graphs, and apply them to each node of the graph. To construct G_n, I claim that the following procedure works:

- Construct graphs $A = G_{n-1}T$ and $B = \overline{G_{n-1}^r}H$.
- Draw an edge from the root of B (i.e. HHH...HH) to HHH...HT.

This is not hard to prove. For each node M in B, applying the operation to M will flip the coins in position $1 + (n - 1 - k) = n - k$. Since the corresponding node in A has its first $n - 1$ coins reversed, this operation is identical, so B is a valid subgraph. Finally, applying the operation to HHH...HH obviously yields HHH...HT, so the claim has been proven.

Obviously G_n is a tree, and furthermore note that HHH...HH is at a distance of n from the root; this is because from there, we simply sweep from the right to the left, flipping each bit. Thus it takes n operations to reach the end of the process. Half of the tree retains its value of $L(C)$, and the other half has $L(C)$ incremented my n. It follows that

$$a_n = a_{n-1} + \frac{1}{2}n = \frac{1}{2}(1 + 2 + \cdots + n) = \frac{1}{4}n(n+1).$$

This completes the proof.

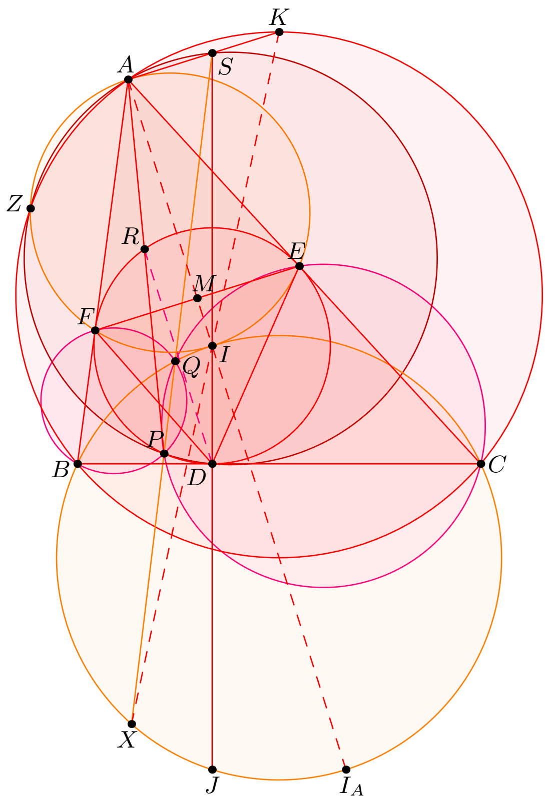
§6 IMO 2019/6 (IND)

Problem 6

Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of $\triangle ABC$ is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to $\overline{EF}$ meets ω at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

First solution, by spiral similarity



Claim 1. $BCIQ$ is cyclic.

Proof. Just notice that

$$\begin{aligned}\measuredangle BQC &= \measuredangle BQP + \measuredangle PQC = \measuredangle BFP + \measuredangle PEC \\ &= \measuredangle FEP + \measuredangle PFE = \measuredangle FPE = \measuredangle FDE = \measuredangle BIC,\end{aligned}$$

as desired. □

Claim 2. $\triangle RFE \sim \triangle IBC$.

Proof. Again this is just angle chasing:

$$\measuredangle RFE = \measuredangle RDE = 90^\circ - \measuredangle DEF = \measuredangle FDI = \measuredangle FBI = \measuredangle IBC,$$

and similarly $\measuredangle FER = \measuredangle BCI$, as desired. □

Let $Z = (ABC) \cap (AEF)$ be the Miquel point of $BCEF$, and let ψ be the spiral similarity at Z sending $\overline{FE}$ to $\overline{BC}$. By Claim 2, ψ sends R to Q. Also let it send D to J, A to K, and P to X. It is not hard to see that J lies on (BIC) with $\overline{IJ} \perp \overline{BC}$ and K is the midpoint of arc BAC on the circumcircle. Furthermore, since $(EF;PR)$ is harmonic, so is $(BC;XI)$; but $\overline{KB}$ and $\overline{KC}$ are tangent to (BIC), so X lies on both (BIC) and $\overline{KI}$.

Let $S = \overline{DJ} \cap \overline{AK}$. It suffices to show that S lies on line PQ.

Claim 3. A, S, D, P, Z are concyclic.

Proof. By the spiral similarity at Z, points A, S, D, Z are already concyclic. But

$$\measuredangle APD = \measuredangle RPD = \measuredangle RDB = \measuredangle(\overline{AI}, \overline{BC}) = \measuredangle(\overline{AS}, \overline{DJ}) = \measuredangle AZD,$$

as desired. $\square$

Claim 4. X, P, Q, S are collinear.

Proof. By the spiral similarity at Z, $\measuredangle ZPX = \measuredangle ZAK = \measuredangle ZAS = \measuredangle ZPS$, so X, P, S are collinear. Finally, notice that $\measuredangle BQX = \measuredangle BCX = \measuredangle FEP = \measuredangle BFP = \measuredangle BQP$. $\square$

Claim 4 is exactly what we want to show, so we are done.

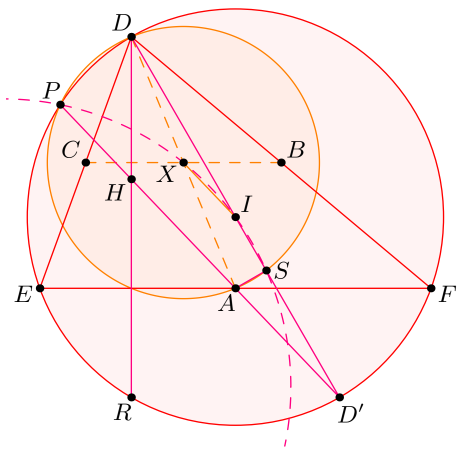
Second solution, by inversion Let Ω be the circumcircle of $\triangle BIC$ and D' the antipode of D on ω. First we claim that $Q \in \Omega$. This follows from

$$\begin{aligned}\measuredangle BQC &= \measuredangle BQP + \measuredangle PQC = \measuredangle BFP + \measuredangle PEC \\ &= \measuredangle FEP + \measuredangle PFE = \measuredangle FPE = \measuredangle FDE = \measuredangle BIC.\end{aligned}$$

Furthermore, the systems $\triangle AFE \cup \omega$ and $\triangle KBC \cup \Omega$ are similar since $\overline{KB}$ and $\overline{KC}$ are tangent to Ω and $\measuredangle FAE = \measuredangle BKC$. Let $\overline{KI}$ intersect Ω again at X. Noting that $\overline{IK}$ is the K-symmedian of $\triangle IBC$, we have

$$\measuredangle BQP = \measuredangle BFP = \measuredangle FRP = \measuredangle FRA = \measuredangle BIK = \measuredangle BIX = \measuredangle BQX,$$

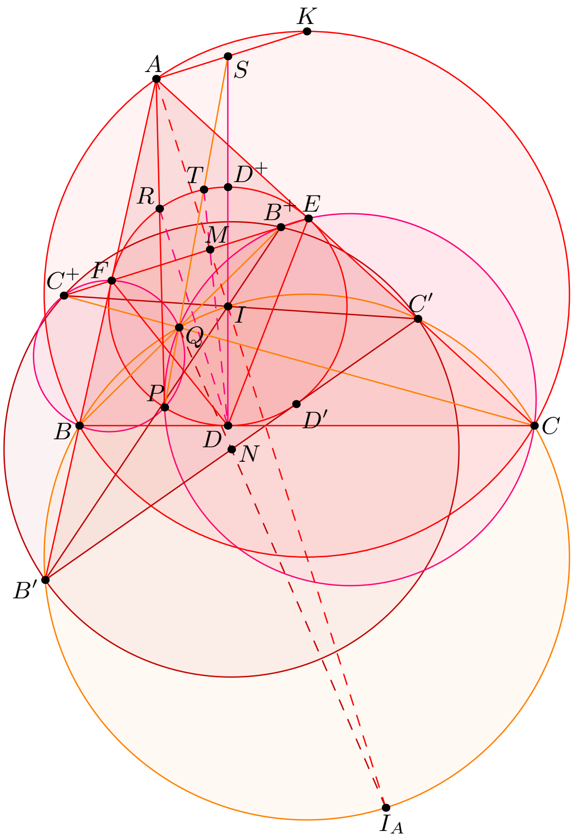
so X, P, Q are collinear. Let $S = \overline{DI} \cap \overline{AK}$. It suffices to show that S lies on line PX.



Invert about ω, sending A, B, C to A', B', C', the midpoints of $\overline{EF}$, $\overline{FD}$, $\overline{DE}$ respectively. Since $BICX$ is harmonic, X is sent to the midpoint of $\overline{B'C'}$ (which also happens to be the midpoint of $\overline{DA'}$). Moreover S is sent to S', the projection of A' onto $\overline{DD'}$. It is sufficient to show that $PX'IS'$ is cyclic. From here on we omit the prime symbol from images of the inversion for readability.

Since $ERFP$ is harmonic, it is well-known that P lies on $\overline{HAD'}$. Finally, P, X, I, S lie on the nine-point circle of $\triangle ADD'$, so we are done.

Third solution, by the Iran Lemma Let I_A be the A-excenter, Ω the circumcircle of $\triangle BIC$, and let $\overline{PQ}$ and $\overline{DI}$ intersect ω again at T and D^+ respectively. Denote by D' the reflection of D over $\overline{AI}$ and M the midpoint of $\overline{EF}$. Let K be the midpoint of arc BAC on the circumcircle, and let Q' lie on Ω such that $\overline{IQ'} \perp \overline{B'C'}$. Denote $B^+ = \overline{BQ} \cap \overline{EF}$ and $C^+ = \overline{CQ} \cap \overline{EF}$, and let $\overline{IB^+}$ and $\overline{IC^+}$ intersect Ω again at B' and C' respectively.



Claim 1. $BCIQ$ is cyclic.

Proof. Just notice that

$$\begin{aligned}\measuredangle BQC &= \measuredangle BQP + \measuredangle PQC = \measuredangle BFP + \measuredangle PEC \\ &= \measuredangle FEP + \measuredangle PFE = \measuredangle FPE = \measuredangle FDE = \measuredangle BIC,\end{aligned}$$

as desired. □

Claim 2. B^+ lies on $(CPQE)$ and C^+ lies on $(BPQF)$.

Proof. $\measuredangle CEB^+ = \measuredangle AEF = \measuredangle CIB = \measuredangle CQB = \measuredangle CQB^+$ and similarly $\measuredangle BFC^+ = \measuredangle BQC^+$. □

Claim 3. B' and C' are the reflections of C and B respectively across $\overline{AI}$.

Proof. By Reim's Theorem on $(BPQFC^+)$ and Ω, $\overline{FC^+} \parallel \overline{B'C}$. Then $\overline{BC'}$ and $\overline{B'C}$ are both perpendicular to $\overline{AI}$, and the desired result follows. $\square$

Claim 4. QIB^+C^+ is cyclic.

Proof. $\measuredangle QIB^+ = \measuredangle QIB' = \measuredangle QBB' = \measuredangle QBF = \measuredangle QC^+F = \measuredangle QC^+B^+$. $\square$

Claim 5. $\overline{I_AQ}$ bisects $\overline{B'C'}$.

Proof. By the Iran Lemma on $\triangle AB'C'$, B^+ and C^+ lie on $(B'C')$. Since Q is the Miquel point of $B'C'C^+B^+$, this is just a well-known result applied to $\triangle IB'C'$: If H denotes the orthocenter of $\triangle IB'C'$, then Q lies on the circle with diameter $\overline{IH}$ (since $H = \overline{B'C^+} \cap \overline{C'B^+}$). But it is well-known that I_A is the reflection of H over the midpoint N of $\overline{B'C'}$. Thus N lies on $\overline{QI_A}$, as desired. $\square$

Notice that

$$\measuredangle I_AB'C' = \measuredangle I_ACC' = \measuredangle I_ACA = \measuredangle BIA = \measuredangle DFE,$$

so $\triangle I_AB'C' \sim \triangle DFE$. Since

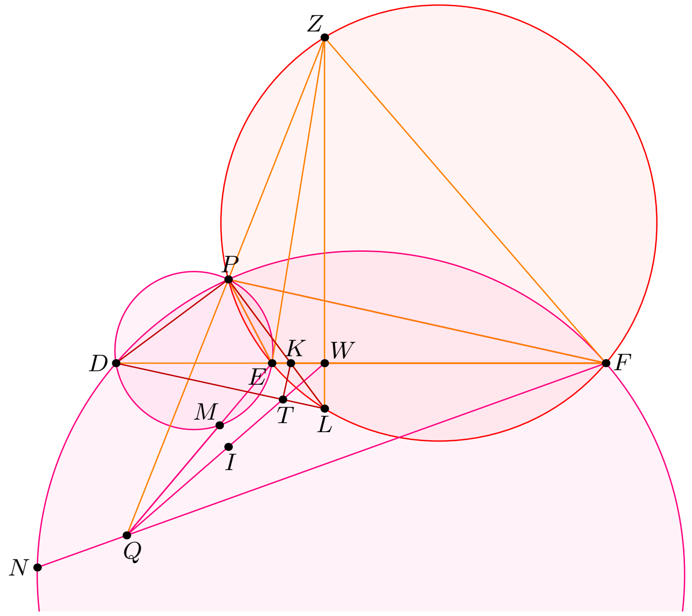
$$\measuredangle B'I_AQ = \measuredangle B'BQ = \measuredangle FBQ = \measuredangle FPQ = \measuredangle FPT$$

but $\overline{I_AQ}$ bisects $\overline{B'C'}$, T lies on $\overline{PQ}$.

By Brocard's Theorem on $DPTD^+$, $S = \overline{DD^+} \cap \overline{PT}$ lies on the polar of M, which is the line through A perpendicular to $\overline{AI}$. This completes the proof.

Fourth solution, by two inversions (Brandon Wang, Luke Robitaille, Michael Ren, Evan Chen) Invert about ω and then invert about P. It is not hard to check that this gives the following problem.

> Let PEF be a triangle and let the P-symmedian meet $\overline{EF}$ and (PEF) at K and L respectively. Let D lie on $\overline{EF}$ with $\angle DPK = 90°$, and let T be the foot from K to $\overline{DL}$. Denote by I the reflection of P about $\overline{EF}$. Finally, let M and N be the harmonic conjugates of P with respect to $\overline{DE}$ and $\overline{DF}$ respectively. Prove that lines EM, FN, TI are concurrent



Since $DPWL$ is cyclic, $\measuredangle ZEP = \measuredangle ZLP = \measuredangle WLP = \measuredangle WDP = \measuredangle EDP$, so $\overline{ZE}$ is tangent to (DPE); similarly, $\overline{ZF}$ is tangent to (DPF).

Note that $\triangle WTP$ is the orthic triangle of $\triangle DKL$, so it is well-known that $\overline{WD}$ bisects $\angle PTK$. However I is the reflection of P over $\overline{WD}$, so W, T, I are collinear.

Finally, $-1 = E(PM;DZ) = F(PN;DZ) = W(PI;DZ)$, so $\overline{EM}$, $\overline{FN}$, $\overline{WI}$ concur on $\overline{PZ}$, as desired.

Fourth solution, by linearity (Edward Wan) Let (BPE) and (CPF) intersect $\overline{BC}$ again at B_1 and C_1 respectively. We start with this familiar claim.

Claim 1. $\triangle RFE \sim \triangle IBC$.

Proof. This is just angle chasing:

$$\measuredangle RFE = \measuredangle RDE = 90^\circ - \measuredangle DEF = \measuredangle FDI = \measuredangle FBI = \measuredangle IBC,$$

and similarly $\measuredangle FER = \measuredangle BCI$, as desired. □

Claim 2. $DB_1 : DC_1 = AC : AB$.

Proof. First $\measuredangle PB_1C_1 = \measuredangle PB_1B = \measuredangle PFB = \measuredangle PEF$, so $\triangle PB_1C_1 \sim \triangle PEF$. But $PERF$ is harmonic, so

$$\frac{PB_1}{PC_1} = \frac{PE}{PF} = \frac{RE}{RF} = \frac{IC}{IB} = \frac{\sin\frac{1}{2}B}{\sin\frac{1}{2}C}.$$

However

$$\begin{aligned}\measuredangle B_1PD &= \measuredangle B_1PF + \measuredangle FPD = \measuredangle B_1BF + \measuredangle FED = \measuredangle DBF + \measuredangle FED \\ &= \measuredangle DIF + \measuredangle FED = 2\measuredangle DEF + \measuredangle FED = \measuredangle DEF,\end{aligned}$$

so $\angle B_1PD = 90^\circ \pm \frac{1}{2}B$ and $\angle C_1PD = 90^\circ \mp \frac{1}{2}C$. Thus by the Ratio Lemma,

$$\frac{DB_1}{DC_1} = \frac{PB_1 \sin\angle B_1PD}{PC_1 \sin\angle C_1PD} = \frac{\sin\frac{1}{2}B\cos\frac{1}{2}B}{\sin\frac{1}{2}C\cos\frac{1}{2}C} = \frac{\sin B}{\sin C} = \frac{AC}{AB},$$

as desired. □

Now define functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ by

$$f_1(\bullet) = \operatorname{Pow}(\bullet, (BI)) - \operatorname{Pow}(\bullet, (CI)),$$
$$f_2(\bullet) = \operatorname{Pow}(\bullet, (BPF)) - \operatorname{Pow}(\bullet, (CPE)).$$

It is well-known that f_1 and f_2 are linear. Also let the external angle bisector of $\angle A$ intersect $\overline{BC}$ at Y and $\overline{DI}$ at S. It suffices to show that $f_2(S) = 0$.

Claim 3. For all points $\bullet$ on $\overline{AS}$, $f_1(\bullet) = f_2(\bullet)$.

Proof. Clearly $f_1(A) = AB \cdot AF - AC \cdot AE = f_2(A)$, so it suffices to show that $f_1(Y) = f_2(Y)$. But this is equivalent to

$$YB \cdot YD - YC \cdot YD = YB \cdot YB_1 \cdot YC \cdot YC_1,$$

or rather $YB \cdot DB_1 = YC \cdot DC_1$. However $YB : YC = AB : AC$ by the Angle Bisector Theorem, so our claim has been reduced to Claim 2. □

Finally S lies on $\overline{DI}$, the radical axis of (BI) and (CI). Hence $f_2(S) = f_1(S) = 0$, and we are done.

Fifth solution, by elliptic curves (yaron235) Instead we prove the following generalization:

> Let ABC be a triangle and let E and F be points on $\overline{AC}$ and $\overline{AB}$ respectively. Let D be a point in the plane and let K be the intersection of the circumcircles of $\triangle DEC$ and $\triangle DFB$. Let S be the intersection point of $\overline{DK}$ and a line through A and parallel to $\overline{EF}$, and let P be the second intersection of the circumcircles of $\triangle ASD$ and $\triangle FED$. Denote by Q the second intersection of the circumcircles of $\triangle PEC$ and $\triangle PFB$. Prove that S lies on $\overline{PQ}$.

It is not hard to check that the problem is a special case of this generalization; the only work that needs to be done is to prove this claim:

Claim 1. In the language of the original problem, $ASDP$ is cyclic.

Proof. Fortunately this is not hard. Just notice that

$$\measuredangle APD = \measuredangle RPD = 90^\circ + \measuredangle EFD + \measuredangle FED = \measuredangle(\overline{AI}, \overline{BC}) = \measuredangle ASD,$$

done. □

Now let's prove the generalization. Work in $\mathbb{C}\mathrm{P}^2$, and let $I = (1 : i : 0)$ and $J = (1 : -i : 0)$ denote the two circular points at infinity, i.e. the two points at infinity common to every circle. Let γ be the locus of points X with the property that S lies on the radical axis of the circumcircles of $\triangle XEC$ and $\triangle XFB$. We only need to prove that P lies on this locus.

Claim 2. γ is an elliptic curve.

Proof. Indeed, if X moves with degree 1, then Y, the intersection of (XEC) and (XFB), moves with degree 2 by homography. Thus the collinearity of points X, Y, S has degree 3, as desired. □

By construction, points S, D, K already lie on γ. Of course points B, F, C, E, I, J lie on γ as well, since we can select arbitrary circles through two points. Also note that for all points X in γ, the second intersection Y of (XEC) and (XFB) also lies on γ, so in the group of γ, $X + Y + C + E + I + J = X + Y + B + F + I + J = 0$. In particular, $C + E = B + F$, so $A = \overline{CE} \cap \overline{BF}$ lies on γ. Also X, Y, S are collinear by definition, so $X + Y + S = 0$ and thus $C + E + I + J = B + F + I + J = S$. Moreover A, C, E are collinear, so $A + C + E = 0$ and $I + J = A + S$. It follows that ∞_{EF}, the point at infinity along $\overline{EF}$, lies on γ.

Recall that we want to prove that P lies on γ. But (DEF) intersects γ again at $-(D + E + F + I + J)$ and (ASD) intersects γ again at $-(A + S + D + I + J)$. It is sufficient to show that these are the same point. But $\infty_{EF} = \overline{EF} \cap \overline{AS}$ lies on γ, so $E + F = A + S$, and we are done.