IMO 2018

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§0 Problems

Problem 1. Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of $\overline{BD}$ and $\overline{CE}$ intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or are the same line.

Problem 2. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \dots, a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+2}$$

for all $i = 1, 2, \dots, n$.

Problem 3. An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + 3 + \ldots + 2018$?

Problem 4. A *site* is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Problem 5. Let $a_1, a_2, \dots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Problem 6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

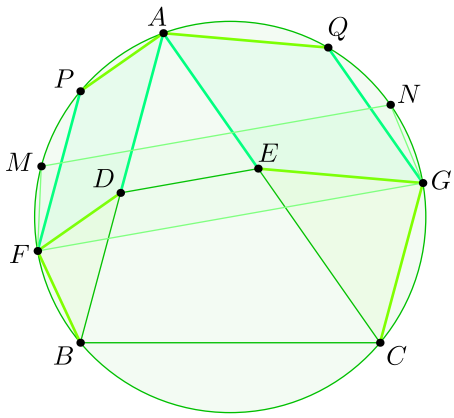
Prove that $\angle BXA + \angle DXC = 180°$.

§1 IMO 2018/1 (HEL)

Problem 1

Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of $\overline{BD}$ and $\overline{CE}$ intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or are the same line.

First solution, by constructing parallelograms Construct points P and Q on Γ such that $ABFP$ and $ACGQ$ are isosceles trapezoids, and let M and N be the midpoints of minor arcs AB and AC respectively. It is obvious that M and N are the midpoints of arcs PF and QG as well. Since $AP = BF = DF$ and $AQ = CG = EG$, $APFD$ and $AQGE$ are parallelograms.



Note that $PF = AD = AE = QG$, so $\widehat{PF} = \widehat{QG}$ and thus $\widehat{MF} = \widehat{NG}$. It follows that $FGNM$ is an isosceles trapezoid, so $\overline{FG} \parallel \overline{MN}$. But both $\overline{DE}$ and $\overline{MN}$ are perpendicular to the internal angle bisector of $\angle A$, so $\overline{DE}$ and $\overline{FG}$ are parallel, as desired.

Second solution, by angle chasing Let $\overline{FD}$ and $\overline{GE}$ intersect Γ again at X and Y respectively. Notice that

$$\measuredangle AXD = \measuredangle AXF = \measuredangle ABF = \measuredangle DBF = \measuredangle FDB = \measuredangle XDA,$$

whence $AX = AD$. Analogously, $AY = AE$, so D, E, X, Y lie on a circle with center A. Finally, by Reim's Theorem, $\overline{DE} \parallel \overline{FG}$, as desired.

§2 IMO 2018/2 (SVK)

Problem 2

Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+2}$$

for all $i = 1, 2, \ldots, n$.

The answer is $3 \mid n$, achieved by the sequence $(2, -1, -1, 2, -1, -1, \ldots)$. I claim that $a_i = a_{i+3}$ for all i, so if $3 \nmid n$, the sequence is constant; this concludes the proof, as $x^2 + 1 = x$ has no real root. Check that

$$\sum_{i=1}^{n} a_{i+2}^2 = \sum_{i=1}^{n} (a_i a_{i+1} a_{i+2} + a_{i+2}) = \sum_{i=1}^{n} (a_i a_{i+1} a_{i+2} + a_i) = \sum_{i=1}^{n} a_i a_{i+3}.$$

This can be rewritten as $0 = \sum_{i=1}^{n} (a_i - a_{i+3})^2$, hence done.

§3 IMO 2018/3 (IRN)

Problem 3

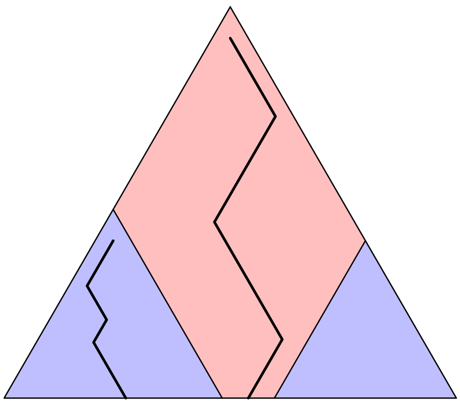
An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + 3 + \ldots + 2018$?

The answer is no. Let $N = 2018$, and assume for the sake of contradiction such a triangle exists. For each number x not at the bottom, let its children be y and z. Draw an arrow from x to $\max(y, z)$, so that if $y > z$, we draw an arrow from x to y, and also $y = x + z$.

Let the chain starting from the top element be a_1, a_2, $\ldots$, a_N (so that a_N is in the bottom row). Since at each step we increment by a different positive integer, it can be shown by induction that $a_i \geq 1 + 2 + \cdots + i$. That is, $a_N \geq 1 + 2 + \cdots + N$. Since every number in the triangle does not exceed $1 + 2 + \cdots + N$, equality holds, and the numbers 1 through 2018 are all adjacent to some number in the chain. Consider the two subtriangles shown below.



We do not include $1 + 2 + \cdots + N$ nor the two adjacent numbers, so neither triangle contains any integer 1 through 2018. By the Pigeonhole Principle one triangle has at least 1008 elements in its bottom row, so the number X at the bottom of the chain from the triangle's topmost element is greater than

$$\begin{aligned} X &\geq (N+1) + (N+2) + \cdots + (N+1008) \\ &= 1008N + 504 \cdot 1009 \\ &= 1009(N+504) - N \\ &> 1009(N+1) \\ &= 1 + 2 + \cdots + N, \end{aligned}$$

a contradiction.

§4 IMO 2018/4 (ARM)

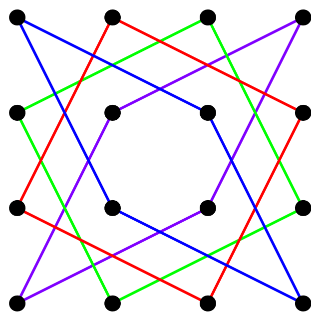
Problem 4

A *site* is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

The answer is 100. To achieve this, checkerboard-color the grid and let Amy take half of the black squares. Now we show that for each 4×4 grid, Amy can place at most four stones if Bob plays optimally. This is clearly sufficient. Consider the following dissection into 4-cycles:



Whenever Amy plays in this 4×4 grid, Bob puts a stone in the opposite vertex in the 4-cycle Amy's stone belongs to. Thus Amy can put at most one stone in each 4-cycle, so we are done.

§5 IMO 2018/5 (MNG)

Problem 5

Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Consider the partial difference

$$x = \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} - \frac{a_n}{a_1} = \frac{a_1 a_n + a_{n+1}^2 - a_n a_{n+1}}{a_1 a_{n+1}}.$$

Take a prime p and analyze the p-adic valuation of the sequence. Assume that for all n, $\nu_p(a_n) \neq \nu_p(a_{n+1})$. This is a valid assumption, because if $\nu_p(a_n) = \nu_p(a_{n+1})$, then $x = 1$, which is an integer.

Since x is always an integer, we have a few cases to consider:

Claim. For all n,

- If $\nu_p(a_n) > \nu_p(a_{n+1})$, then $\nu_p(a_{n+1}) \geq \nu_p(a_1)$.
- If $\nu_p(a_n) < \nu_p(a_{n+1})$, then $\nu_p(a_{n+1}) = \nu_p(a_1)$.

Proof. The first case is trivial. Just note that $\nu_p(a_n/a_{n+1}) \geq 0$, so $a_1 \mid a_{n+1} - a_n$. Now, if $\nu_p(a_n) < \nu_p(a_{n+1})$, then a_n/a_{n+1} will not be an integer, whence $\nu_p(a_{n+1}) \leq \nu_p(a_1)$. However if $\nu_p(a_{n+1}) < \nu_p(a_1)$, then a_n/a_{n+1} and $(a_{n+1} - a_n)/a_1$ cannot sum to an integer, contradiction. $\square$

It is now obvious that the sequence defined by $\nu_p(a_n)$ will eventually converge, and there can be a jump in the sequence of ν_p's (the second case above) only if $p \mid a_1$, which occurs for finitely many p. Thus the sequence is eventually constant.

§6 IMO 2018/6 (POL)

Problem 6

A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

First solution, by inversion We first require the following two lemmas.

Lemma 1

If two quadrilaterals have the same angles and both obey $AB \cdot CD = BC \cdot DA$, then they are similar.

Proof. Omitted. □

Lemma 2

If point S in quadrilateral $ABCD$ has a isogonal conjugate S^*, then $\angle BSA + \angle DSC = 180^\circ$.

Proof. Let $P = \overline{AB} \cap \overline{CD}$ and $Q = \overline{AD} \cap \overline{BC}$, and denote by W, X, Y, Z the projections of S onto $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively. Note that S and S^* are isogonal conjugates with respect to the four triangles $\triangle PAD$, $\triangle PBC$, $\triangle QAB$, and $\triangle QDC$. Since the center of the pedal circle of S is the midpoint of $\overline{SS^*}$, points W, X, Y, Z lie on the pedal circle of S.

Now, all that remains is an angle chase:

$$\begin{aligned}\measuredangle BSA + \measuredangle DSC &= \measuredangle BSW + \measuredangle WSA + \measuredangle DSY + \measuredangle YSC \\ &= \measuredangle BXW + \measuredangle WZA + \measuredangle DZY + \measuredangle YXC \\ &= \measuredangle WZY + \measuredangle YXW = 0^\circ,\end{aligned}$$

as desired. □

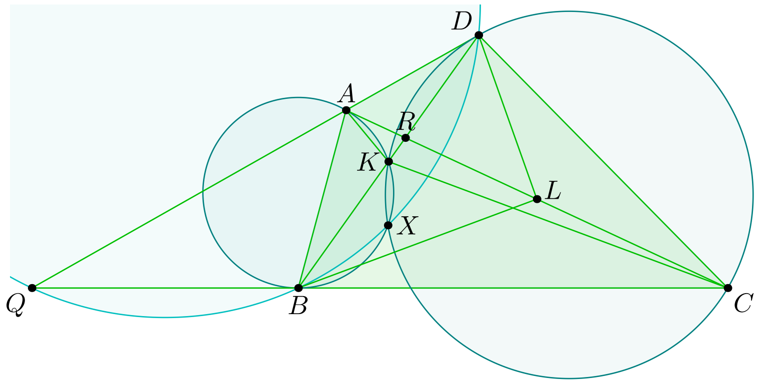
Now, invert about X with arbitrary radius r, denoting the inverse of T by T'. Notice that $\measuredangle XB'A' = -\measuredangle XAB = -\measuredangle XCD = \measuredangle XD'C'$, and similarly $\measuredangle XC'B' = \measuredangle XA'D'$. Furthermore by the Inversion Distance Formula,

$$A'B' \cdot C'D' = \frac{r^2 \cdot AB}{XA \cdot XB} \cdot \frac{r^2 \cdot CD}{XC \cdot XD} = \frac{r^2 \cdot BC}{XB \cdot XC} \cdot \frac{r^2 \cdot DA}{XD \cdot XA} = B'C' \cdot D'A'.$$

We can also check that

$$\measuredangle D'A'B' = \measuredangle D'A'X + \measuredangle XA'B' = \measuredangle XDA + \measuredangle ABX = \measuredangle XBC + \measuredangle ABX = \measuredangle ABC,$$

and analogously we find by Lemma 1 that $D'A'B'C' \sim ABCD$. Transforming $D'A'B'C'$ back to $ABCD$, X is mapped to its isogonal conjugate, so by Lemma 2, $\angle BXA + \angle DXC = 180^\circ$, and we are done.



Second solution, by angle chasing Let $Q = \overline{AD} \cap \overline{BC}$. Since $AB/AD = CB/CD$, there exists a point E on $\overline{BD}$ such that $\overline{AE}$ bisects $\angle DAB$ and $\overline{CE}$ bisects $\angle BCD$. Thus there exists a point K on $\overline{BD}$ with $\measuredangle CAB = \measuredangle DAK$ and $\measuredangle BCA = \measuredangle KCD$. Let the circumcircles of $\triangle AKB$ and $\triangle CKD$ intersect at X. I claim that X is the desired point. First, we prove a key claim.

Claim. $\overline{BD}$ bisects $\angle AKC$.

Proof. Notice that

$$\frac{KA}{KD} = \frac{\sin\angle BDA}{\sin\angle KAD} = \frac{\sin\angle BDA}{\sin\angle BAC} \quad\text{and}\quad \frac{KC}{KD} = \frac{\sin\angle BDC}{\sin\angle KCD} = \frac{\sin\angle BDC}{\sin\angle BCA}.$$

By the ratio lemma,

$$\frac{KA}{KC} = \frac{\sin\angle BDA}{\sin\angle BDC}\cdot\frac{\sin\angle BCA}{\sin\angle BAC} = \frac{RA}{RC}\cdot\frac{DC}{DA}\cdot\frac{BA}{BC} = \frac{RA}{RC},$$

and the desired result readily follows. □

Notice that by the claim, $\measuredangle BXA + \measuredangle DXC = \measuredangle DXA + \measuredangle BXC = \measuredangle DKA + \measuredangle BKC = 0^\circ$, so it is sufficient to show that $\measuredangle XBC = \measuredangle XDA$ (and the other case follows analogously). But

$$\measuredangle BXD = \measuredangle BXK + \measuredangle KXD = \measuredangle BAK + \measuredangle KCD = \measuredangle CAD + \measuredangle BCA = \measuredangle CQA,$$

so $BQDX$ is cyclic and $\measuredangle XBC = \measuredangle XBQ = \measuredangle XDQ = \measuredangle XDA$, as desired.