IMO 2017

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§0 Problems

Problem 1. For each integer $a_0 > 1$, define the sequence a_0, a_1, a_2 by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \quad \text{for each } n \geq 0.$$

Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

Problem 3. A hunter and an invisible rabbit play a game in the plane. The rabbit and hunter start at points $A_0 = B_0$. In the n^{th} round of the game ($n \geq 1$), three things occur in order:

(i) The rabbit moves invisibly from A_{n-1} to a point A_n such that $A_{n-1}A_n = 1$.

(ii) The hunter has a tracking device (e.g. dog) which reports an approximate location P_n of the rabbit, such that $P_nA_n \leq 1$.

(iii) The hunter moves visibly from B_{n-1} to a point B_n such that $B_{n-1}B_n = 1$.

Let $N = 10^9$. Can the hunter guarantee that $A_NB_N < 100$?

Problem 4. Let R and S be different points on a circle Ω such that $\overline{RS}$ is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of $\overline{RT}$. Point J is chosen on minor arc RS of Ω so that the circumcircle Γ of $\triangle JST$ intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that line KT is tangent to Γ.

Problem 5. An integer $N \geq 2$ is given. A collection of $N(N + 1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N - 1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold: no one stands between the two tallest players, no one stands between the third and fourth tallest players, ..., no one stands between the two shortest players. Show that this is always possible.

Problem 6. An ordered pair (x, y) of integers is a *primitive point* if $\gcd(x, y) = 1$. Given a finite set S of primitive points, prove that there exist a positive integer n and integers a_0, a_1, ..., a_n such that for each $(x, y) \in S$, we have

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

§1 IMO 2017/1 (SAF)

Problem 1

For each integer $a_0 > 1$, define the sequence a_0, a_1, a_2 by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \qquad \text{for each } n \geq 0.$$

Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

The answer is $3 \mid a_0$. The proof is easy, but rigorizing it is substantially harder. Here is one among many ways of doing so.

Proof of sufficiency: Let m be the minimum value attained by the sequence. If $3 \mid a_0$, then all elements of the sequence are divisible by 3.

I claim $m = 3$. Indeed, if $m \geq 6$, then $(m-3)^2 \geq m$, so the sequence reaches k^2 for some $k \leq m-3$ and subsequently $k \leq m-3 < m$. This contradicts the minimality of m, as required.

Then the sequence repeats $3 \to 6 \to 9 \to 3 \to \cdots$, end proof.

Proof of necessity: Firstly if $a_k \equiv 2 \pmod 3$ for some k, then $a_{k+i} = a_k + 3i$ for all i since 2 is not a quadratic residue modulo 3; thus the sequence is unbounded and the problem does not hold. Let m be the minimum value attained by the sequence.

I claim $m \equiv 2 \pmod 3$, which proves the problem. Assume for contradiction $m \equiv 1 \pmod 3$; then $m \geq 4$ implies $(m-2)^2 \geq m$, so the sequence reaches k^2 for some $k \leq (m-2)^2$ and subsequently $k \leq m - 2 < m$. This contradicts the minimality of m, as desired.

§2 IMO 2017/2 (ALB)

Problem 2

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

The answer is $f \equiv 0$, $f(x) = x - 1$, $f(x) = 1 - x$, which all clearly work. Let $P(x, y)$ denote the assertion. First note that $P(0, 0)$ gives $f(f(0)^2) = 0$. In what follows we assume $f \not\equiv 0$. We may assume that $f(0) > 0$ since:

- if $f(0) = 0$, then $P(x, 0)$ gives $f(x) = 0$ for all x;
- if f obeys the functional equation, then so does $-f$.

The task is to show $f(x) = 1 - x$.

Claim 1. $f(z) = 0 \iff z = 1$, and $f(0) = 1$, $f(1) = 0$.

Proof. For all z,

$$P\left(z, \frac{z}{1-z}\right) \implies f\left(f(z)f\left(\frac{z}{z-1}\right)\right) = 0.$$

If $z \neq 1$ and $f(z) = 0$, then $f(0) = 0$, contradiction, thus $f(z) = 0 \implies z = 1$. But $f(f(0)^2) = 0$, so f attains 0 — thus we must have $f(1) = 0$.

Furthermore $f(0)^2 = 1$, but since $f(0) > 0$, it follows that $f(0) = 1$. $\square$

Claim 2. f is injective.

Proof. Note that $P(x, 1)$ gives $1 + f(x+1) = f(x)$, so if $f(a) = f(b)$, then for all nonnegative integers C, $f(a + C) = f(b + C)$. Choose C sufficiently large so that there exist x and y with

$$x + y = a + C + 1 \quad \text{and} \quad xy = b + C;$$

therefore $f(x + y) = f(xy) - 1$.

Then $P(x, y)$ gives $f(f(x)f(y)) = 1$, so $f(x)f(y) = 0$ and $1 \in \{x, y\}$. This implies $a = b$, as claimed. $\square$

Finally, $P(x, 0)$ gives $f(f(x)) + f(x) = 1$, so

$$f(x) = 1 - f(f(x)) = f(f(f(x))) = f(1 - f(x)).$$

From this $f(x) = 1 - x$, and we are done.

§3 IMO 2017/3 (AUT)

Problem 3

A hunter and an invisible rabbit play a game in the plane. The rabbit and hunter start at points $A_0 = B_0$. In the n^{th} round of the game ($n \geq 1$), three things occur in order:

(i) The rabbit moves invisibly from A_{n-1} to a point A_n such that $A_{n-1}A_n = 1$.

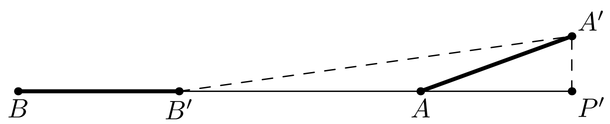
(ii) The hunter has a tracking device (e.g. dog) which reports an approximate location P_n of the rabbit, such that $P_nA_n \leq 1$.

(iii) The hunter moves visibly from B_{n-1} to a point B_n such that $B_{n-1}B_n = 1$.

Let $N = 10^9$. Can the hunter guarantee that $A_NB_N < 100$?

The answer is no. Consider the following process, occurring over m rounds. For convenience, we will refer to this as an m-move.

- The rabbit moves from point A to the point A' such that $AA' = m$, one unit each round, and if P' is the projection of A' onto line AB, then $A'P' = 1$.
- During each round, let the dog report the projection of the rabbit's location onto line AB. This is always possible since the distance from the rabbit to line AB never exceeds 1.
- Because the hunter doesn't know which half-plane the rabbit is on, he must move along $\overline{AB}$. Otherwise, it is possible that the rabbit is on the other side of the line as he moves, and he moves farther away from the rabbit.
- For convenience, assume the hunter is magically able to deduce the exact location of the rabbit at the end of the process.



Claim. Suppose that at some point in time, the rabbit is at point A and the hunter is at point B, and furthermore $AB = x$. After a 101-move, if the rabbit moves to A' and the hunter moves to B', then the rabbit can guarentee that $A'B' \geq \sqrt{x^2 + 1/101}$.

Proof. Recall that $A'P' = 1$, whence $B'P' = x + \sqrt{m^2-1} - m$. It follows that

$$\begin{aligned} A'B'^2 &\geq 1 + \left(x + \sqrt{m^2-1} - m\right)^2 \\ &= 1 + x^2 + (m^2 - 1) + m^2 + 2x\sqrt{m^2-1} - 2m\sqrt{m^2-1} - 2xm \\ &= x^2 + 2m^2 - 2xm + 2x\sqrt{m^2-1} - 2m\sqrt{m^2-1} \\ &= x^2 + 2(m-x)\left(m - \sqrt{m^2-1}\right). \end{aligned}$$

Let $m = 101$ above. Check that

$$A'B'^2 \geq x^2 + \frac{2(m-x)}{m + \sqrt{m^2-1}} \geq x^2 + \frac{m-x}{m} \geq x^2 + \frac{1}{101},$$

as desired. $\square$

Finally, we can perform a 101-move $\lfloor 10^9/101 \rfloor$ times; that is,

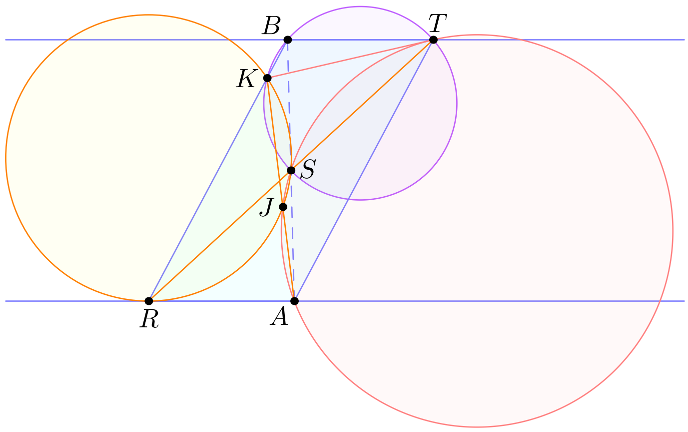
$$A_N B_N^2 \geq \left\lfloor \frac{10^9}{101} \right\rfloor \cdot \frac{1}{101} \geq \frac{10^9 - 100}{101^2} > 100^2,$$

so $A_N B_N > 100$, as desired.

§4 IMO 2017/4 (LUX)

Problem 4

Let R and S be different points on a circle Ω such that $\overline{RS}$ is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of $\overline{RT}$. Point J is chosen on minor arc RS of Ω so that the circumcircle Γ of $\triangle JST$ intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that line KT is tangent to Γ.



Let B be the reflection of A over S, so that $ARBT$ is a parallelogram. By Reim's theorem on Ω and Γ, we have $\overline{RK} \parallel \overline{TA}$, so R, K, B are collinear. By the converse of Reim's theorem with Ω, we have $TBKS$ cyclic, so

$$\measuredangle KTA = \measuredangle TKB = \measuredangle TSB = \measuredangle TSA,$$

as desired.

§5 IMO 2017/5 (RUS)

Problem 5

An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold: no one stands between the two tallest players, no one stands between the third and fourth tallest players, $\ldots$, no one stands between the two shortest players. Show that this is always possible.

Let S_k denote the set of soccer players with height h obeying $(k-1)(N+1) < h \leq k(N+1)$, so that S_1, S_2, $\ldots$, S_N is a partition of the soccer players into N groups of size $N+1$. We will prove inductively it is possible to select two people from each set S_k.

Scan the line from left to right until we find two people in the same set S_k. Then

- select those two people;
- delete everyone scanned; and
- delete everyone in the set S_k.

Besides S_k, in all groups N people remain, and the pair we have selected is the leftmost one, so we may induct down.

Remark. The problem is invariant when swapping location and height, so essentially all solutions are isomorphic to each other. The solution works if we instead group the players by location and select two players from the group whose second-shortest player is as short as possible.

§6 IMO 2017/6 (USA)

Problem 6

An ordered pair (x, y) of integers is a *primitive point* if $\gcd(x, y) = 1$. Given a finite set S of primitive points, prove that there exist a positive integer n and integers a_0, a_1, $\ldots$, a_n such that for each $(x, y) \in S$, we have

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

In English, we want to prove that for any set of irreducible lattice points, there is a nonconstant homogeneous polynomial $P \in \mathbb{Z}[x, y]$ such that $P(x, y) = 1$ for all $(x, y) \in S$.

We induct on $|S|$, with the base case $|S| = 1$ being Bézout's lemma. Assume that for some set $S = \{(x_1, y_1), \ldots, (x_{k-1}, y_{k-1})\}$ of irreducible lattice points, we have $P(x_i, y_i)$ for $i = 1, \ldots, k-1$. The goal is to prove for each choice of $(x_k, y_k) \notin S$, there is a polynomial Q such that $Q(x_i, y_i)$ for $i = 1, \ldots, k$.

Let $n = \deg P$ and let $ax_k + by_k = 1$, where a, b exist by Bézout's lemma. We will exhibit a polynomial of the form

$$Q(x, y) = P(x, y)^U + C \cdot (ax + by)^V \prod_{i=1}^{k-1} (y_i \cdot x - x_i \cdot y),$$

where A is an integer and $nU = V + (k - 1)$.

By design, $Q(x_i, y_i) = 1$ for $i = 1, \ldots, k-1$, so it will suffice to find U, C such that $Q(x_k, y_k) = 1$.

If $y_ix_k - x_iy_k = 0$ for some i, then since $(x_k, y_k) \neq (x_i, y_i)$, we have $x_k = -x_i$, $y_k = -y_i$. Then take $M = 2$ and win. Henceforth $y_ix_k - x_iy_k \neq 0$ for all i.

Claim. $P(x_k, y_k)$ and $y_ix_k - x_iy_k$ are relatively prime for all $i = 1, \ldots, k - 1$.

Proof. Assume $p \mid y_ix_k - x_iy_k$. If $p \mid x_k$, then $p \nmid y_k$, so apply the below argument with x, y swapped. In what follows, $p \nmid x_k$ (and thus $p \nmid x_i$).

Then we have $y_k/x_k \equiv y_i/x_i \pmod p$, so

$$\begin{aligned} P(x_k, y_k) &\equiv \sum_{j=0}^{n} a_jx_k^{n-j}y_k^j \equiv x_k^n \sum_{j=0}^{n} a_j \left(\frac{y_k}{x_k}\right)^j \equiv x_k^n \sum_{j=0}^{n} a_j \left(\frac{y_i}{x_i}\right)^j \\ &\equiv \left(\frac{x_k}{x_i}\right)^n P(x_i, y_i) \equiv \left(\frac{x_k}{x_i}\right)^n \not\equiv 0 \pmod p, \end{aligned}$$

which proves the claim. $\square$

Since $ax_k + by_k = 1$, we have $P(x_k, y_k)$ and $(ax_k + by_k)^v \prod_{i=1}^{k-1}(y_ix_k - x_iy_k)$ are relatively prime. Take U a large multiple of $2\varphi\left(\prod_{i=1}^{i-1}(y_ix_k - x_iy_k)\right)$, so that $nU \geq k - 1$ and there is an integer C for which $Q(x_k, y_k) = 1$. This completes the proof.