ELMO 2021

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Last updated June 20, 2021

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§0 Problems

Problem 1. Let ABC be a triangle, and let P and Q lie on sides AB and AC such that the circumcircle of $\triangle APQ$ is tangent to segment BC at a point D. Let E lie on segment BC such that $BD = EC$. Line DP intersects the circumcircle of $\triangle CDQ$ again at X, and line DQ intersects the circumcircle of $\triangle BDP$ again at Y. Prove that points D, E, X, Y are concyclic.

Problem 2. Let $n \geq 2$ be an integer and let a_1, a_2, $\ldots$, a_n be integers such that $n \mid a_i - i$ for all integers $1 \leq i \leq n$. Prove there exists an infinite sequence b_1, b_2, $\ldots$ with $b_i \in \{a_1, a_2, \ldots, a_n\}$ for each i, such that

$$\sum_{i=1}^{\infty} \frac{b_i}{n^i} \in \mathbb{Z}.$$

Problem 3. Each cell of a 100×100 grid is colored with one of 101 colors. A cell is *diverse* if, among the 199 cells in its row and column, every color appears at least once. Determine the maxmum possible number of diverse cells.

Problem 4. Suppose the set of positive integers is partitioned into $n \geq 2$ disjoint arithmetic progressions S_1, S_2, $\ldots$, S_n with common differences d_1, d_2, $\ldots$, d_n. Prove that there exists exactly one index $1 \leq i \leq n$ such that

$$\prod_{j \neq i} d_j \in S_i.$$

Problem 5. Let n and k be positive integers. Two infinite sequences (s_i) and (t_i) are *equivalent* if $s_i = s_j$ if and only if $t_i = t_j$ for all positive integers i and j, and a sequence (t_i) has *equi-period* k if t_1, t_2, $\ldots$ and t_{k+1}, t_{k+2}, $\ldots$ are equivalent. In terms of n and k, how many sequences of equi-period k are there in the set of sequences with each entry in the set $\{1, 2, \ldots, n\}$, up to equivalence?

Problem 6. In triangle ABC, points D, E, F lie on segments BC, CA, AB, respectively, such that each of the quadrilaterals $AFDE$, $BDEF$, $CEFD$ has an incircle. Prove that the inradius of $\triangle ABC$ is twice the inradius of $\triangle DEF$.

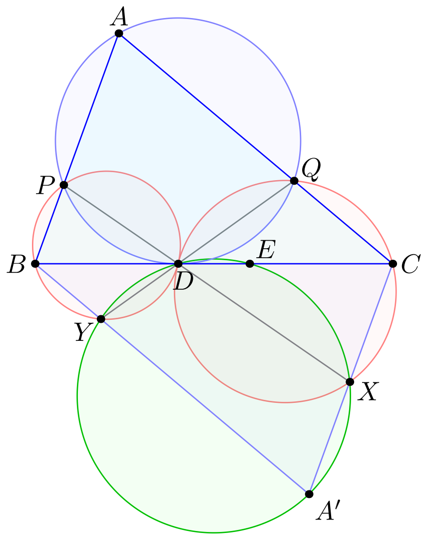
§1 ELMO 2021/1 (Eric Shen)

Problem 1 (ELMO 2021/1)

Let ABC be a triangle, and let P and Q lie on sides AB and AC such that the circumcircle of $\triangle APQ$ is tangent to segment BC at a point D. Let E lie on segment BC such that $BD = EC$. Line DP intersects the circumcircle of $\triangle CDQ$ again at X, and line DQ intersects the circumcircle of $\triangle BDP$ again at Y. Prove that points D, E, X, Y are concyclic.

We present two solutions.

First solution, by angle chasing Since $\measuredangle BYD = \measuredangle BPD = \measuredangle APD = \measuredangle AQD$, we have $\overline{BY} \parallel \overline{AC}$ and analogously $\overline{CX} \parallel \overline{AB}$. Construct $A' = \overline{BY} \cap \overline{CX}$, so $ABA'C$ is a parallelogram.



Note that $\measuredangle XDY = \measuredangle PQD = \measuredangle PAQ = \measuredangle XA'Y$, so D, X, Y, A' are concyclic. Now observe that $\measuredangle A'ED = \measuredangle ADE = \measuredangle APD = \measuredangle BPD = \measuredangle BYD = \measuredangle A'YD$, so E lies on $(DXA'Y)$ as well.

Second solution, by circumcenters (Pitchayut Saengrungkongka) Let O, O_B, O_C, S be the centers of (APQ), (BDP), (CDQ), (DXY), respectively. Clearly, OO_BSO_C is parallelogram. The rest is just projection chasing: let $\operatorname{proj}(U)$ denote the projection of U onto $\overline{BC}$. We have

$$\begin{aligned}\operatorname{proj}(S) &= \operatorname{proj}(O_B) + \operatorname{proj}(O_C) - \operatorname{proj}(O) \\ &= \frac{B+D}{2} + \frac{C+D}{2} - D \\ &= \frac{B+C}{2},\end{aligned}$$

so S lies on the perpendicular bisector of $\overline{BC}$, which coincides with the perpendicular bisector of $\overline{DE}$.

§2 ELMO 2021/2 (Maxim Li)

Problem 2 (ELMO 2021/2)

Let $n \geq 2$ be an integer and let a_1, a_2, $\ldots$, a_n be integers such that $n \mid a_i - i$ for all integers $1 \leq i \leq n$. Prove there exists an infinite sequence b_1, b_2, $\ldots$ with $b_i \in \{a_1, a_2, \ldots, a_n\}$ for each i, such that

$$\sum_{i=1}^{\infty} \frac{b_i}{n^i} \in \mathbb{Z}.$$

Evidently we may select integers z_0, z_1, z_2, $\ldots$ all in $\{a_1, a_2, \ldots, a_n\}$ so that

$$z_0 + z_1 n + \cdots + z^{k-1} n^{k-1} \equiv 0 \pmod{n^k}$$

for each k. Thus each of the numbers

$$\frac{z_0}{n}, \quad \frac{z_1}{n} + \frac{z_0}{n^2}, \quad \frac{z_2}{n} + \frac{z_1}{n^2} + \frac{z_0}{n^3}, \quad \ldots$$

are integers.

Now since these integers are bounded, we may find k and ℓ with

$$\frac{z_{k-1}}{n} + \cdots + \frac{z_0}{n^k} = \frac{z_{k+\ell-1}}{n} + \cdots + \frac{z_0}{n^{k+\ell}} =: A.$$

This implies that

$$\frac{z_{k+\ell-1}}{n} + \cdots + \frac{z_k}{n^\ell} = \left(1 - \frac{1}{n^\ell}\right) A.$$

Finally, let $(b_1, b_2, \ldots) = (z_{k+\ell-1}, \ldots, z_0, z_{k+\ell-1}, \ldots, z_0, \ldots)$, so that

$$\frac{b_1}{n} + \frac{b_2}{n^2} + \cdots = \frac{\frac{z_{k+\ell-1}}{n} + \cdots + \frac{z_k}{n^\ell}}{1 - \frac{1}{n^\ell}} = \frac{z^{k-1}}{n} + \cdots + \frac{z_0}{n^\ell} = A,$$

which is an integer.

§3 ELMO 2021/3 (Maxim Li)

Problem 3 (ELMO 2021/3)

Each cell of a 100×100 grid is colored with one of 101 colors. A cell is *diverse* if, among the 199 cells in its row and column, every color appears at least once. Determine the maxmum possible number of diverse cells.

The answer is $100^2 - 4$, achieved by generalizing the following construction:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \\ 4 & 4 & 4 & \color{red}{5} & \color{red}{6} \\ 6 & 6 & 6 & \color{red}{6} & \color{red}{5} \end{bmatrix}.$$

Now we prove that this is maximal. If some color appears at most 98 times, then we may find two rows and two columns without this color, so we may find four non-diverse squares.

Henceforth all colors appear at least 99 times, so we may assume without loss of generality the colors, 1, 2, ..., 100, 101 each appear 99, 99, ..., 99, 100 times.

For each $k = 1, \dots, 100$, select a color c_k that does not appear in row k, and let x_k be the number of columns containing c_k. If $x_k \neq 100$ for all k, there is a non-diverse square in every row, so at least 100 non-diverse squares.

Otherwise $x_k = 100$ for some k, implying $c_k = 101$. Thus the color 101 does not appear in every row, so repeating the argument with columns instead of rows gives the required bound.

§4 ELMO 2021/4 (Brandon Wang)

Problem 4 (ELMO 2021/4)

Suppose the set of positive integers is partitioned into $n \geq 2$ disjoint arithmetic progressions S_1, S_2, $\ldots$, S_n with common differences d_1, d_2, $\ldots$, d_n. Prove that there exists exactly one index $1 \leq i \leq n$ such that

$$\prod_{j \neq i} d_j \in S_i.$$

Evidently every sequence is a complete residue class. Exactly one sequence contains

$$\sum_i \prod_{j \neq i} d_j,$$

and that is the sequence that works.

§5 ELMO 2021/5 (Sean Li)

Problem 5 (ELMO 2021/5)

Let n and k be positive integers. Two infinite sequences (s_i) and (t_i) are *equivalent* if $s_i = s_j$ if and only if $t_i = t_j$ for all positive integers i and j, and a sequence (t_i) has *equi-period* k if t_1, t_2, $\dots$ and t_{k+1}, t_{k+2}, $\dots$ are equivalent. In terms of n and k, how many sequences of equi-period k are there in the set of sequences with each entry in the set $\{1, 2, \dots, n\}$, up to equivalence?

The answer is n^k.

We'll construct a bijection between $(a_1, \dots, a_k) \in [n]^k$ and equivalence classes of sequences (s_i) of equi-period k.

First direction: Suppose we are given $(a_1, \dots, a_k) \in [n]^k$. Define a counter $C = 0$. For each $i = 1, 2, \dots, k$:

- If $a_i \le n - C$, then let $s_i = C + 1$, $s_{k+i} = C + 2$, $s_{2k+i} = C + 3$, $\dots$, $s_{(a-1)k+i} = C + a_1$, and increment C by a_i.
- Otherwise let $s_i = n+1-a_i \le C$, which will determine s_{k+i}, s_{2k+i}, $\dots$ through previously-known cycles using numbers $\le C$.

Second direction: Suppose we are given (s_i). Keep a running counter $C = 0$, that we will use to number off the distinct values of observe. For each $i = 1, 2, \dots, k$:

- If all of s_i, s_{k+i}, s_{2k+i}, $\dots$ are greater than C (i.e. we've never seen them before), then let a_i be the period of this sequence. Without loss of generality (up to equivalence) we may let $s_i = C + 1$, $s_{k+i} = C + 2$, $\dots$, $s_{(a_i-1)k+i} = C + a_i$, and increment C by a_i.
- If any of s_i, s_{k+i}, s_{2k+i}, $\dots$ is $\le C$, then all of them are determined by a previously-known cycle and each term $\le C$, so we may let $a_i = n + 1 - s_i$.

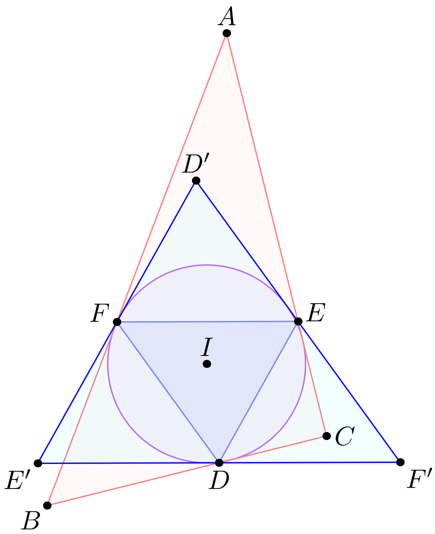
It is easy to see that the above two operations are inverses of each other, so we have established the desired bijection.

§6 ELMO 2021/6 (Maxim Li)

Problem 6 (ELMO 2021/6)

In triangle ABC, points D, E, F lie on segments BC, CA, AB, respectively, such that each of the quadrilaterals $AFDE$, $BDEF$, $CEFD$ has an incircle. Prove that the inradius of $\triangle ABC$ is twice the inradius of $\triangle DEF$.

Consider the anticomplementary triangle $D'E'F'$ of $\triangle DEF$. We will show that the incircles of $\triangle ABC$ and $\triangle D'E'F'$ coincide.



First note that $AF - AE = DF - DE = D'E - D'F$, so there is a circle ω_A tangent to rays AE, AF, $D'E$, $D'F$. Define ω_B and ω_C analogously.

Assume for contradiction the circles ω_A, ω_B, ω_C do not coincide. One can check that the pairwise exsimilicenters of ω_A, ω_B, ω_C are $\overline{BC} \cap \overline{E'F'} = D$, $\overline{CA} \cap \overline{F'D'} = E$, $\overline{AB} \cap \overline{D'E'} = F$. By Monge's theorem, points D, E, F are collinear, contradiction.