2018 Mock AIME Solutions

by TheUltimate123

(Remastered on March 16, 2019)

Instructions

1. DO NOT BEGIN READING THE PROBLEMS UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor penalties for wrong answers.
3. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam will require the use of a calculator.
4. A combination of your AIME score and your American Mathematics Contest 12 score is *not* used to determine eligibility for participation in the nonexistent Mock USA Mathematical Olympiad (USAMO). A combination of your AIME score and your American Mathematics Contest 10 score is *not* used to determine eligibility for participation in the nonexistent USA Junior Mathematical Olympiad (USAJMO).
5. PM your answers to TheUltimate123 on AoPS.

1 Problems

1. The ASCII value of a digit is 48 more than the digit. For instance, the digit 0 has a value of 48, while the digit 7 has a value of 55. Let $g(n)$ be defined as the sum of the ASCII values of the digits of n for all positive integers n, when expressed in base 10. For instance, $g(10) = 97$ and $g(1234) = 202$. The sum of all positive integers n such that $g(n) = n$ is N. Find the remainder when N is divided by 1000.

2. There exist two non-intersecting circles with radii 4 and 3. Suppose the length of their common internal tangent is 21. Then, the length of their common external tangent is $\sqrt{d}$, where d is a positive integer. Find d.

3. Twelve points are chosen uniformly and at random on the circumference of a circle. The probability there exists a diameter $\overline{MN}$ such that all twelve points lie on the same side of $\overline{MN}$ is $\frac{p}{q}$ for relatively prime integers p and q. Find $p + q$.

4. For all subsets S of the set $\{1, 2, 3, \ldots, 511\}$, let $p(S)$ be the product of all the elements of S, with $p(\emptyset)$ defined as 1, and $v(S)$ be the largest integer such that $2^{v(S)}$ divides $p(S)$. Over all such S, find the expected value of $v(S)$.

5. Denote by S the set of real numbers x such that $0 \le x \le 10000$. Let $f : S \to \mathbb{R}$ be a function such that
$$100f(xy) = f(x+y)f(x-y) + x + y$$
for all x, y such that both sides are defined. Find the remainder when $f(2018)^2$ is divided by 1000.

6. In triangle $\triangle ABC$, X lies on $\overline{AB}$ and Y lies on $\overline{AC}$ such that $\overline{BY}$ bisects $\angle ABC$ and $\overline{CX}$ bisects $\angle ACB$. $\overline{BY}$ and $\overline{CX}$ intersect at a point P. Suppose that P lies on the circumcircle of triangle $\triangle AXY$. If $AX = 15$ and $AY = 24$, find AP^2.

7. Let $a_0 = 20$, $a_1 = 18$, and $a_n = a_{n-1} + a_{n-2}$ for all integers $n \ge 2$. There exist relatively prime positive integers m and n such that
$$\sum_{i=0}^{\infty} \frac{a_i}{3^i} = \frac{m}{n}.$$
Find $m + n$.

8. The base of a cone with radius 1 and height $\sqrt{3}$ lies on the horizontal. A plane p passes through the cone and forms an angle of 30 degrees with the horizontal. The intersection of the horizontal and p is a line that is tangent to the base. The area of the cross section formed when p passes through the cone is $N\pi$, and N^2 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. An ant is initially positioned at the origin. Every second, it moves 1 unit in a direction that is chosen uniformly and at random. Suppose that for all nonnegative integers n, E_n is the ant's expected distance from the origin. If K denotes the smallest positive integer such that $E_K > 2018$, find the remainder when K is divided by 1000.

10. Find the remainder when
$$\sum_{n=0}^{100} (9^n + 11^n).$$
is divided by 1000.

11. Find the smallest positive integer n such that $n^3 + 5n^2 + 2n + 2$ is divisible by 1000.

12. In triangle $\triangle ABC$, $AB = 13$, $BC = 14$, $CA = 15$, and a point P lies on $\overline{BC}$. Let Q be the foot of the perpendicular from P to $\overline{AB}$ and R be the foot of the perpendicular from P to $\overline{AC}$. Suppose I_B and I_C are the incenters of triangles $\triangle PBQ$ and $\triangle PCR$, respectively. Then the maximum possible area of $\triangle PI_BI_C$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. There are N ordered quadruples of positive integers (a, b, c, d) that satisfy $a + b + 2c + 3d = 89$. Find the remainder when N is divided by 1000.

14. Suppose 2019 chicks are sitting in a circle. Suddenly, each chick randomly pecks either the chick on its left or the chick on its right with equal probability. Let k be the number of chicks that were not pecked. The probability k is odd can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

15. Let a, b, c, and d be positive real numbers such that

$$195 = a^2 + b^2 = c^2 + d^2 = \frac{13\left(ac + bd\right)^2}{13b^2 - 10bc + 13c^2} = \frac{5\left(ad + bc\right)^2}{5a^2 - 8ac + 5c^2}.$$

Find the greatest integer that does not exceed $a + b + c + d$.

2 Answers

1. 545
2. 489
3. 515
4. 251
5. 982
6. 507
7. 179
8. 011
9. 325
10. 102
11. 483
12. 457
13. 353
14. 537
15. 034

3 Solutions

1. The ASCII value of a digit is 48 more than the digit. For instance, the digit 0 has a value of 48, while the digit 7 has a value of 55. Let $g(n)$ be defined as the sum of the ASCII values of the digits of n for all positive integers n, when expressed in base 10. For instance, $g(10) = 97$ and $g(1234) = 202$. The sum of all positive integers n such that $g(n) = n$ is N. Find the remainder when N is divided by 1000.

   Solution. $\boxed{545}$. First note that a one-digit integer cannot satisfy the condition, as $g(n) \geq 48$. In addition, integers with $k \geq 4$ digits cannot satisfy the condition as well, as $g(n) \leq (48+9)k = 57k < n$. Now, we only need to consider two-digit and three-digit integers.

   If n has two digits, let $n = \overline{ab}$. Then, $g(n) = (48+a) + (48+b) = 96 + a + b$. Also, $n = 10a + b$, so $96 + a + b = 10a + b$, so $9a = 96$. This is clearly not possible.

   If n has three digits, let $n = \overline{abc}$. Then, $g(n) = (48+a) + (48+b) + (48+c) = 144 + a + b + c$. Also, $n = 100a + 10b + c$. Thus, $144 + a + b + c = 100a + 10b + c$, so $144 = 99a + 9b$, and $9(11a + b) = 144 \implies 11a + b = 16$. The only pair (a, b) that satisfies this is $(1, 5)$. Since c can be any digit, the desired sum is $150 + 151 + \cdots + 159 = 309 \cdot 5 = 1545$, and the requested remainder is 545.

2. There exist two non-intersecting circles with radii 4 and 3. Suppose the length of their common internal tangent is 21. Then, the length of their common external tangent is $\sqrt{d}$, where d is a positive integer. Find d.

   Solution. $\boxed{489}$. Suppose the center of the circle with radius 4 is P and the center of the circle with radius 3 is Q. Suppose the common internal tangent touches circle P at A and circle Q at B. Let $\overline{PQ}$ intersect $\overline{AB}$ at X. Then, by AA, $\triangle PAX \sim \triangle QBX$. It follows that $\frac{AX}{BX} = \frac{AP}{BQ} = \frac{4}{3}$. Since $AX + BX = 21$, $AX = 12$ and $BX = 9$. It follows that $PX = 4\sqrt{10}$ and $QX = 3\sqrt{10}$, so $PQ = 7\sqrt{10}$.

   Suppose the common external tangent touches circle P at A' and Q at B'. Suppose C lies on $\overline{A'P}$ such that $\overline{A'P} \perp \overline{CQ}$. Then, since $\overline{A'P} \parallel \overline{B'Q}$, $CP = 1$ and $AC = 3$. Note that since $\overline{CQ} \parallel \overline{A'B'}$, $CQ = A'B'$. It is easy to see that $CQ = \sqrt{PQ^2 - CP^2} = \sqrt{489}$, and the answer is 489.

3. Twelve points are chosen uniformly and at random on the circumference of a circle. The probability there exists a diameter $\overline{MN}$ such that all twelve points lie on the same side of $\overline{MN}$ is $\frac{p}{q}$ for relatively prime integers p and q. Find $p + q$.

   Solution. $\boxed{515}$. Label the points $A_1, A_2, \ldots, A_{12}$. Assume A_1 is the leftmost point. Because $\overline{MN}$ is the diameter of the circle, we want on the other points to be in the same semicircle. This occurs with probability $\frac{1}{2^{11}}$, because there are 11 other points. But, to select A_1, we have 12 options, so the probability in question is $\frac{12}{2^{11}} = \frac{3 \cdot 2^2}{2^{11}} = \frac{3}{2^9} = \frac{3}{512}$, and the answer is $3 + 512 = 515$.

4. For all subsets S of the set $\{1, 2, 3, \ldots, 511\}$, let $p(S)$ be the product of all the elements of S, with $p(\emptyset)$ defined as 1, and $v(S)$ be the largest integer such that $2^{v(S)}$ divides $p(S)$. Over all such S, find the expected value of $v(S)$.

   Solution. $\boxed{251}$. Since each number has a $\frac{1}{2}$ chance of being in S, the value of $v_2(p(S))$ is simply $\frac{1}{2}\left(v_2(1) + v_2(2) + \ldots + v_2(511)\right)$. Let $a_n = v_2(1) + v_2(2) + \ldots + v_2(2^n)$. We seek $\frac{1}{2}(a_9 - v_2(512)) = \frac{1}{2}(a_9 - 9)$. Notice that $a_0 = 0$, and $a_n = 2a_{n-1} + 1$. The former is obvious, but to show the latter, observe that the sequence $v_2(n)$ is $0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, \ldots$ like a ruler. Note that the sum of the last 8 terms is one more that the sum of the first 8, because the only difference is the final term. It is easy to see by induction that $a_n = 2^n - 1$. Therefore, the answer is $\frac{1}{2}(511-9) = \frac{502}{2} = 251$.

5. *(e_power_pi_times_i)* Denote by S the set of real numbers x such that $0 \le x \le 10000$. Let $f : S \to \mathbb{R}$ be a function such that

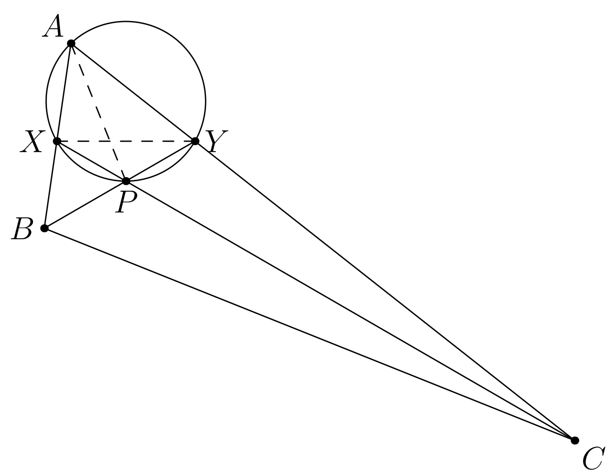
$$100f(xy) = f(x+y)f(x-y) + x + y$$

for all x, y such that both sides are defined. Find the remainder when $f(2018)^2$ is divided by 1000.

Solution. $\boxed{982}$. Plugging in $(0,0)$ results in $100f(0) = f(0)^2$. Either $f(0) = 0, 100$. Then plug in $(x,0)$ to get $100f(0) = f(x)^2 + x$, so if $f(0) = 0$ the function only outputs imaginary numbers for positive x. Then $100^2 = f(x)^2 + x$, so $f(2019)^2 = 100^2 - 2018 = 7982$.

6. In triangle $\triangle ABC$, X lies on $\overline{AB}$ and Y lies on $\overline{AC}$ such that $\overline{BY}$ bisects $\angle ABC$ and $\overline{CX}$ bisects $\angle ACB$. $\overline{BY}$ and $\overline{CX}$ intersect at a point P. Suppose that P lies on the circumcircle of triangle $\triangle AXY$. If $AX = 15$ and $AY = 24$, find AP^2.

Solution. $\boxed{507}$.



Notice that

$$180 = \angle A + \angle XPY = \angle A + \angle BPC = \angle A + \left(90 + \frac{\angle A}{2}\right) = 90 + \frac{3\angle A}{2},$$

whence $\angle A = 60$.

Since P is the incenter of $\triangle ABC$, $\overline{AP}$ is the angle bisector of $\angle A$. Then,

$$\angle PAX = \angle PAY = \angle PXY = \angle PYX = 30^\circ.$$

By the Law of Cosines on $\triangle AXY$, $XY = 21$. Since $\triangle PXY$ is an isosceles triangle with an angle of 120°, $PX = PY = 7\sqrt{3}$. Then, by Ptolemy's Theorem on quadrilateral $AXPY$,

$$24 \cdot 7\sqrt{3} + 15 \cdot 7\sqrt{3} = 21 \cdot AP \implies AP = 13\sqrt{3},$$

so $AP^2 = 507$.

7. Let $a_0 = 20$, $a_1 = 18$, and $a_n = a_{n-1} + a_{n-2}$ for all integers $n \geq 2$. There exist relatively prime positive integers m and n such that

$$\sum_{i=0}^{\infty} \frac{a_i}{3^i} = \frac{m}{n}.$$

Find $m + n$.

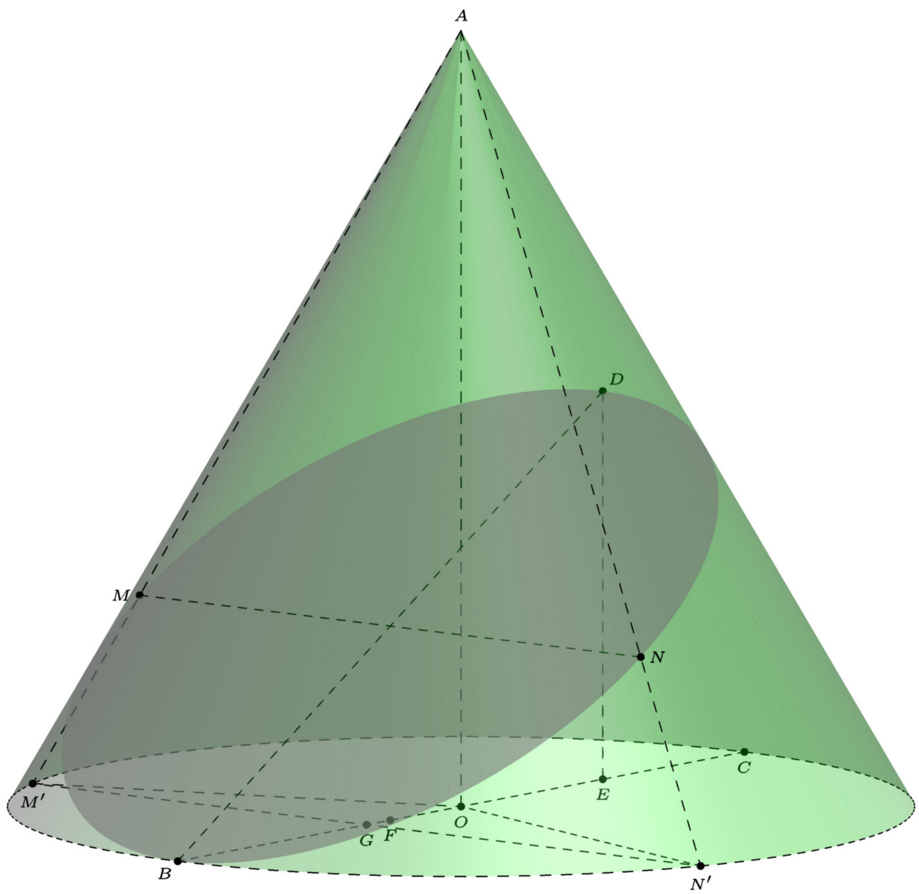
Solution. $\boxed{179}$. Let the expression be S. Note that

$$\begin{aligned}
S &= \sum_{i=0}^{\infty} \frac{a_i}{3^i} \\
&= \frac{a_0}{3^0} + \frac{a_1}{3^1} + \sum_{i=2}^{\infty} \frac{a_i}{3^i} \\
&= 26 + \sum_{i=2}^{\infty} \frac{a_i}{3^i} \\
&= 26 + \sum_{i=2}^{\infty} \frac{a_{i-1} + a_{i-2}}{3^i} \\
&= 26 + \sum_{i=2}^{\infty} \frac{a_{i-1}}{3^i} + \sum_{i=2}^{\infty} \frac{a_{i-2}}{3^i} \\
&= 26 + \frac{1}{3} \sum_{i=2}^{\infty} \frac{a_{i-1}}{3^{i-1}} + \frac{1}{9} \sum_{i=2}^{\infty} \frac{a_{i-2}}{3^{i-2}} \\
&= 26 + \frac{1}{3} \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{9} \sum_{i=0}^{\infty} \frac{a_i}{3^i} \\
&= 26 - \frac{20}{3} + \frac{1}{3}\left(\frac{a_0}{3^0}\right) + \frac{1}{3} \sum_{i=1}^{\infty} \frac{a_i}{3^i} + \frac{1}{9} \sum_{i=0}^{\infty} \frac{a_i}{3^i} \\
&= \frac{58}{3} + \frac{1}{3} \sum_{i=0}^{\infty} \frac{a_i}{3^i} + \frac{1}{9} \sum_{i=0}^{\infty} \frac{a_i}{3^i} \\
&= \frac{58}{3} + \frac{4}{9} S. \\
\implies \frac{5}{9} S = \frac{58}{3} &\implies S = \frac{174}{5} \implies m + n = 179,
\end{aligned}$$

and we are done.

8. *(dilworthpenguins)* The base of a cone with radius 1 and height $\sqrt{3}$ lies on the horizontal. A plane p passes through the cone and forms an angle of 30 degrees with the horizontal. The intersection of the horizontal and p is a line that is tangent to the base. The area of the cross section formed when p passes through the cone is $N\pi$, and N^2 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. $\boxed{011}$. Note that by definition, the cross section is an ellipse.



Let the vertex of the cone be A, and B be the point of tangency between the base of the cone and the intersection of plane p with the horizontal (which we will call l). Suppose C lies on the base such that $\overline{BC}$ is a diameter. Let D be the midpoint of $\overline{AC}$ and O be the center of the base. Let E be the foot of the perpendicular from D to $\overline{BC}$, and F be the midpoint of $\overline{BE}$. Let $\overline{MN}$ be the minor axis of the ellipse. Let M' be the intersection of line AM with the horizontal. Similarly define N'. Suppose the intersection of $\overline{BC}$ and $\overline{M'N'}$ is G. Let P and Q lie on the circumference of the base such that $\overline{PQ}$ lies directly under $\overline{MN}$, with $PM < PN$ and $QN < QM$.

Note that M' and N' lie on the circumference of the base, since M and N lie on the surface of the cone. It is easy to see that $\triangle ABC$ is an equilateral triangle with side length 2. Then, $\overline{BD}$ is the major axis of the ellipse. In addition, it is easy to see that $AD = \sqrt{3}$, so the semi-major axis has length $\frac{\sqrt{3}}{2}$. It suffices to find MN.

Consider the distances of points from the horizontal. Call the distance from a point X to the horizontal z_X. Then, $z_A = \sqrt{3}$, and thus $z_D = \frac{\sqrt{3}}{2}$. Note that $z_M = z_N = \frac{\sqrt{3}}{4}$, and $z_{M'} = z_{N'} = 0$. Since $\triangle AMN \sim \triangle AM'N'$, $MN = \frac{3}{4}M'N'$.

Since $BE = \frac{3}{2}$, $BF = \frac{3}{4}$, and $OF = \frac{1}{4}$. Note that F lies on $\overline{PQ}$. It follows by the similarity between $\triangle AMN$ and $\triangle AM'N'$ that $OG = \frac{4}{3}OF = \frac{1}{3}$. Since $OM' = ON' = 1$, $GM' = GN' = \frac{2\sqrt{2}}{3}$, and $M'N' = \frac{4\sqrt{2}}{3}$. It follows that $MN = \sqrt{2}$, and the semi-minor axis of the ellipse is $\frac{\sqrt{2}}{2}$.

Then, the area of the ellipse is $\pi \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4}\pi$. It follows that $N^2 = \frac{3}{8}$, and the answer is $3 + 8 = 11$, and we are done.

9. An ant is initially positioned at the origin. Every second, it moves 1 unit in a direction that is chosen uniformly and at random. Suppose that for all nonnegative integers n, E_n is the ant's expected distance from the origin. If K denotes the smallest positive integer such that $E_K > 2018$, find the remainder when K is divided by 1000.

Solution. $\boxed{325}$. Note that $E_0 = 0$. Suppose that when moving from time $n-1$ to time n, the ant moves 1 unit in the direction that forms an angle of θ_n with the line connecting the origin and the point at which the ant was. Then, we have

$$E_n^2 = E_{n-1}^2 + 1 - 2E_{n-1}\cos\theta_n.$$

Notice that the probability the ant moves at an angle of θ_n is equal to the probability the ant moves at an angle of $180 - \theta_n$, and that $\cos\theta_n + \cos(180 - \theta_n) = 0$. It follows that the expected value of $\cos\theta_n$ is 0, and by Linearity of Expectation,

$$E_n^2 = E_{n-1}^2 + 1.$$

Since $E_0 = 0$, $E_n = \sqrt{n}$ by induction. Because after K seconds we expect the ant to be more than 2018 units from the origin, $E_K > 2018$, so $K > 2018^2$, and $K^2 = 2018^2 + 1 \equiv 18^2 + 1 = 325 \pmod{1000}$, and we are done.

10. *(tigerche)* Find the remainder when

$$\sum_{n=0}^{100} (9^n + 11^n).$$

is divided by 1000.

Solution. $\boxed{102}$. Let

$$a_n = 9^n + 11^n = (10-1)^n + (10+1)^n.$$

Since we are dealing with modulo 10^3, we are encouraged to expand this with the Binomial Theorem. Modulo 10^3, we get

$$a_n \equiv \begin{cases} 100n^2 - 100n + 2, & \text{if } n \text{ is even;} \\ 20n, & \text{if } n \text{ is odd.} \end{cases} \pmod{1000}$$

We then have

$$\begin{aligned}\sum_{n=0}^{100} a_n &\equiv 100\left(2^2 + 4^2 + \cdots + 100^2\right) - 100(2 + 4 + \cdots + 100) + 20(1 + 3 + \cdots + 99) + 2 \cdot 51 \\ &\equiv 100 \cdot 4 \cdot \frac{50 \cdot 51 \cdot 101}{6} - 100 \cdot 50 \cdot 51 + 20 \cdot 50^2 + 102 \\ &\equiv 102, \quad \pmod{1000}\end{aligned}$$

the answer.

11. *(e_power_pi_times_i)* Find the smallest positive integer n such that $n^3 + 5n^2 + 2n + 2$ is divisible by 1000.

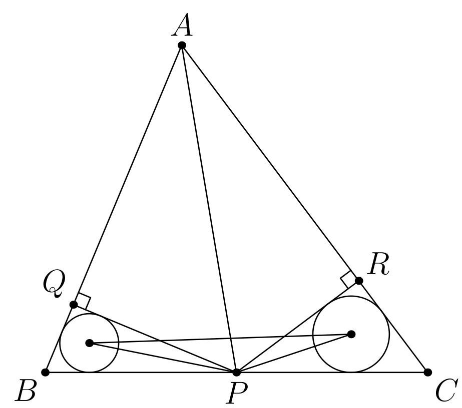
Solution. $\boxed{483}$. By inspection, $n \equiv 3 \pmod 8$. Finding n modulo 125 would be quite tedious. Instead, we will find n mod 5, and "lift" the modulus to 125. By inspection, $n \equiv 1, 3 \pmod 5$. We can split this into two cases: $n \equiv 1$ and $n \equiv 3$ modulo 5.

- *Case 1:* $n \equiv 1 \pmod 5$. Then, let $n \equiv 5u+1 \pmod{25}$. It follows that $(5u+1)^3 + 5(5u+1)^2 + 2(5u+1) + 2 \equiv 0 \pmod{25}$. Notice that when we expand, many terms are divisible by 25. It follows that $15u + 1 + 5 + 10u + 2 + 2 \equiv 0$, or $10 \equiv 0 \pmod{25}$. This is obviously false, so there are no solutions for $n \equiv 1 \pmod 5$.
- *Case 2:* $n \equiv 3 \pmod 5$. Then, let $n \equiv 5u+3 \pmod{25}$. It follows that $(5u+3)^3 + 5(5u+3)^2 + 2(5u+3) + 2 \equiv 0 \pmod{25}$. Expanding gives $135u + 27 + 45 + 10u + 6 + 2 \equiv 0$, or $20u + 5 \equiv 0 \pmod{25}$. Dividing by 5 gives $4u + 1 \equiv 0 \pmod 5$, or $u \equiv 1 \pmod 5$.

It follows that $n \equiv 8 \pmod{25}$. Now we can let $n \equiv 25v + 8$. It follows that $(25v+8)^3 + 5(25v+8)^2 + 2(25v+8) + 2 \equiv 0 \pmod{125}$. Expanding gives $4800v + 512 + 320 + 50v + 16 + 2 \equiv 0$, or $100v + 100 \equiv 0 \pmod{125}$. Dividing by 25 gives $4v + 4 \equiv 0 \pmod 5$, or $v \equiv 4 \pmod 5$. It follows that $n \equiv 108 \pmod{125}$. Since $n \equiv 3 \pmod 8$, the Chinese Remainder Theorem gives $n \equiv 483 \pmod{1000}$, and we are done.

12. In triangle $\triangle ABC$, $AB = 13$, $BC = 14$, $CA = 15$, and a point P lies on $\overline{BC}$. Let Q be the foot of the perpendicular from P to $\overline{AB}$ and R be the foot of the perpendicular from P to $\overline{AC}$. Suppose I_B and I_C are the incenters of triangles $\triangle PBQ$ and $\triangle PCR$, respectively. Then the maximum possible area of $\triangle PI_BI_C$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. $\boxed{457}$.



First, we will find a closed form for the area of $\triangle PI_BI_C$. Notice that $\overline{PI_B}$ bisects $\angle BPQ$ and $\overline{PI_C}$ bisects $\angle CPR$, so

$$\angle I_BPI_C = 180 - \frac{(90 - \angle B) + (90 - \angle C)}{2} = 180 - \frac{\angle A}{2}$$

Since $\angle I_BPI_C$ is fixed, it suffices to maximize $PI_B \cdot PI_C$.

Consider $\triangle PBI_B$. Notice that $\angle BI_BP = 90 + \frac{\angle BQP}{2} = 135°$. By the Law of Sines,

$$\frac{PI_B}{\sin \angle PBI_B} = \frac{BP}{\sin \angle BI_BP} \implies PI_B = \sqrt{2} \cdot BP \cdot \sin\left(\frac{\angle B}{2}\right).$$

Similarly, we know $PI_C = \sqrt{2} \cdot CP \cdot \sin\left(\frac{\angle C}{2}\right)$. It follows that

$$\begin{aligned}
[PI_BI_C] &= \frac{1}{2} \cdot PI_B \cdot PI_C \cdot \sin \angle I_BPI_C \\
&= \frac{1}{2} \cdot \sqrt{2} \cdot BP \cdot \sin\left(\frac{\angle B}{2}\right) \cdot \sqrt{2} \cdot CP \cdot \sin\left(\frac{\angle C}{2}\right) \cdot \sin\left(\frac{\angle A}{2}\right) \\
&= BP \cdot CP \cdot \sin\left(\frac{\angle A}{2}\right) \sin\left(\frac{\angle B}{2}\right) \sin\left(\frac{\angle C}{2}\right).
\end{aligned}$$

Since the angles are fixed, it suffices to maximize $BP \cdot CP$. Since $BP + CP = 14$, by AM-GM the maximum possible value of $BP \cdot CP$ occurs when $BP = CP = 7$.

It follows that

$$[PI_BI_C] = 49 \sin\left(\frac{\angle A}{2}\right) \sin\left(\frac{\angle B}{2}\right) \sin\left(\frac{\angle C}{2}\right)$$

Suppose X is the foot of the perpendicular from A to $\overline{BC}$. It is well known that $AX = 12$, $BX = 5$, and $CX = 9$. It is easy to see by the Cosine Addition Formula that $\cos \angle A = \frac{33}{65}$. Then, we can apply the Half Angle Formulas to see that

$$\begin{aligned}
[PI_BI_C] &= 49\sqrt{\frac{\left(1 - \frac{33}{65}\right)\left(1 - \frac{5}{13}\right)\left(1 - \frac{3}{5}\right)}{2 \cdot 2 \cdot 2}} \\
&= 49\sqrt{\frac{\frac{32}{65} \cdot \frac{8}{13} \cdot \frac{2}{5}}{2 \cdot 2 \cdot 2}} \\
&= 49\sqrt{\frac{8^2}{65^2}} = \frac{49 \cdot 8}{65} = \frac{392}{65}.
\end{aligned}$$

The answer is then $392 + 65 = 457$.

13. There are N ordered quadruples of positive integers (a, b, c, d) that satisfy $a + b + 2c + 3d = 89$. Find the remainder when N is divided by 1000.

Solution. $\boxed{353}$. It is nicer to deal with nonnegative integers. Suppose $w = a - 1$, $x = b - 1$, $y = c - 1$, and $z = d - 1$. Then, $w + x + 2y + 3z = 82$. We will attempt to find a function that returns the number of quadruples for which $w + x + 2y + 3z$ is equal to some value by recursion.

Suppose $P_1(n)$ is the number of solutions where $x = y = z = 0$. It is easy to see that

$$P_1(n) = 1.$$

Similarly, suppose $P_2(n)$ is the number of solutions where $y = z = 0$. Then, $w + x = n$. It is easy to see that

$$P_2(n) = \sum_{i=0}^{n} P_1(i) = n + 1.$$

Suppose $P_3(n)$ is the number of solutions where $z = 0$. This case is more tricky, because the coefficient of y in our Diophantine Equation is 2, not 1. We can split into two cases: $n = 2m$ and $n = 2m - 1$. If $n = 2m$, then

$$P_3(2m) = \sum_{i=0}^{m} P_2(2i) = \sum_{i=1}^{m+1} 2i - 1 = (m+1)^2 = \frac{(n+2)^2}{4}.$$

If $n = 2m - 1$, then

$$P_3(2m-1) = \sum_{i=1}^{m} P(2i-1) = \sum_{i=1}^{m} 2i = m(m+1) = \frac{(n+2)^2 - 1}{4}.$$

Finally, we have a total of $2 \cdot 3 = 6$ cases for $P_4(n)$, as we need to consider different values modulo 6.

However, we seek $P_4(82) = P_4(6 \cdot 13 + 4)$. Therefore, we only need to consider $P_4(6m+4)$. We have

$$\begin{aligned} P_4(6m+4) &= P_3(1) + P_3(4) + \cdots + P_3(6m+1) + P_3(6m+4) \\ &= \sum_{i=0}^{m} P_3(6i+1) + \sum_{i=0}^{m} P_3(6i+4) \\ &= \frac{1}{4}\left(\left(\sum_{i=1}^{2m+2}(3i)^2\right) - (m+1)\right) \\ &= \frac{1}{4}\left(\frac{3(2m+2)(2m+3)(4m+5)}{2} - (m+1)\right) \\ P_4(6 \cdot 13 + 4) &= \frac{1}{4}\left(\frac{3 \cdot 28 \cdot 29 \cdot 57}{2} - 14\right) \\ &= \frac{1}{4}\left(14 \cdot (3 \cdot 29 \cdot 57 - 1)\right) \\ &= \frac{7}{2} \cdot 4958 \\ &= 7 \cdot 2479 \equiv 7 \cdot 479 \equiv 353 \pmod{1000}, \end{aligned}$$

and we are done.

14. Suppose 2019 chicks are sitting in a circle. Suddenly, each chick randomly pecks either the chick on its left or the chick on its right with equal probability. Let k be the number of chicks that were not pecked. The probability k is odd can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

Solution. $\boxed{537}$. Define $p_n(k)$ as the probability k chicks were not pecked if n chicks were sitting in the circle, where n is odd. We claim

$$p_n(k) = \frac{1}{2^{n-1}}\binom{n}{2k}.$$

Proof. Number the chicks 1 to n. Suppose we have a circular string s composed of L's and R's such that a L represents a peck to the left and a R represents a peck to the right. (Note circular means that the character after the last character is the first character). A chick is not pecked iff the letter to the left is L and the letter to the right is R. Now, reorder the string s to a new circular string s' such that the chicks are in the order $1, 3, 5, \ldots, n, 2, 4, 6, \ldots, n-1$.

The number of unpecked chicks is then the number of times the string "LR" appears in s'. Note that if "LR" appears k times, "RL" also appears k times. This is because s' is circular, so there has to be a point where the string transitions from R's to L's. Note that this reasoning is similar to the concept of prefix sums. In addition, the "LR's" and "RL's" are alternating.

We can choose s' in 2^n ways. Now, we can choose the starting positions of the "LR's" and "RL's" in $\binom{n}{2k}$ ways. Note that there is no overcount because even if we choose two adjacent positions, we can construct a string such as "LRL." Consider the first position we choose in the string. It can be either "LR" or "RL," so we must multiply by 2. Note that the rest of the positions we choose rely on the previous, so they are determined by what we choose for the first. Therefore,

$$p_n(k) = \frac{2}{2^n}\binom{n}{2k} = \frac{1}{2^{n-1}}\binom{n}{2k},$$

and our claim has been proven. ■

Since for $p_n(k)$ to be nonzero, $k \leq \lfloor \frac{n}{2} \rfloor$, the probability k is odd is

$$\sum_{i=0}^{505} p_{2019}(2n-1) = \frac{1}{2^{2018}} \left(\sum_{i=0}^{505} \binom{2019}{4n-2} \right) = \frac{m}{2^{2018}}$$

for some m. We can evaluate m by Roots of Unity Filter. Consider the function $f(n) = (n+1)^{2019}$. Suppose $\omega = e^{\pi i/2} = i$ is a 4^{th} root of unity. Then,

$$m = \frac{f(1) - f(\omega) + f(\omega^2) - f(\omega^3)}{4} = \frac{2^{2019} - (1+i)^{2019} + 0^{2019} - (1-i)^{2019}}{4}.$$

Suppose $t = (1+i)^{2019} + (1-i)^{2019}$. Note that $m = \dfrac{2^{2019} - t}{4}$. Writing in Polar Form,

$$\begin{aligned} t &= \left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^{2019} + \left(\sqrt{2}e^{\frac{7\pi i}{4}}\right)^{2019} \\ &= 2^{\frac{2019}{2}} \left(e^{\frac{2019\pi i}{4}} + e^{\frac{2019 \cdot 7\pi i}{4}} \right) \\ &= 2^{\frac{2019}{2}} \left(e^{\frac{3\pi i}{4}} + e^{\frac{5\pi i}{4}} \right) \\ &= 2^{\frac{2019}{2}} \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \right) \\ &= -2^{\frac{2019}{2}}\sqrt{2} \\ &= -2^{1010}. \end{aligned}$$

It follows that

$$m = \frac{2^{2019} + 2^{1010}}{4} = 2^{2017} + 2^{1008} = 2^{1008}\left(2^{1009} + 1\right).$$

Then, our desired probability is

$$\frac{m}{2^{2018}} = \frac{2^{1008}\left(2^{1009}+1\right)}{2^{2018}} = \frac{2^{1009}+1}{2^{1010}}.$$

It is easy to see that $\gcd\left(2^{1009}+1, 2^{1010}\right) = 1$, so we desire $2^{1009} + 1 + 2^{1010} = 3 \cdot 2^{1009} + 1$. Let this value be x. By CRT, we only seek x modulo 8 and 125.

It is easy to see that $x \equiv 1 \pmod{8}$. By Euler's Formula, $2^{\phi(125)} \equiv 2^{100} \equiv 1 \pmod{125}$. Therefore, $x \equiv 3 \cdot 2^9 + 1 \equiv 3 \cdot 512 + 1 \equiv 37 \pmod{125}$. We can write $x = 8j + 1$ for some j. Then, $8j + 1 \equiv 37 \pmod{125}$, and $8j \equiv 36 \pmod{125}$, so $2j \equiv 9$ and $j \equiv \frac{9}{2} \equiv \frac{567}{126} \equiv 67 \pmod{125}$. It follows that $x \equiv 8 \cdot 67 + 1 \equiv 537 \pmod{1000}$, and we are done.

15. Let a, b, c, and d be positive real numbers such that

$$195 = a^2 + b^2 = c^2 + d^2 = \frac{13\left(ac+bd\right)^2}{13b^2 - 10bc + 13c^2} = \frac{5\left(ad+bc\right)^2}{5a^2 - 8ac + 5c^2}.$$

Find the greatest integer that does not exceed $a + b + c + d$.

Solution. $\boxed{034}$. Clearly, an Algebra Bash would be catastrophic. Instead, we present a geometric approach.

Lemma. *In cyclic quadrilateral $ABCD$ with circumradius R, the area K of $ABCD$ is*

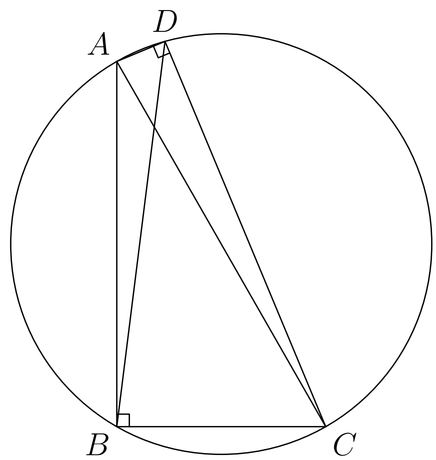
$$K = 2R^2 \sin A \sin B \sin \theta,$$

where θ is either one of the angles formed by the diagonals AC and BD of quadrilateral $ABCD$.

Proof. Suppose $\widehat{AB} = w$, $\widehat{BC} = x$, $\widehat{CD} = y$, and $\widehat{DA} = z$. By the Extended Law of Sines, $BD = 2R \sin A$ and $AC = 2R \sin B$. It follows that

$$K = \frac{1}{2} \cdot BD \cdot AC \cdot \sin \theta = 2R^2 \sin A \sin B \sin \theta,$$

and our lemma has been proven. ■



Consider a cyclic quadrilateral $ABCD$ with $\angle ABC = \angle CDA = 90^\circ$ and $AB = a$, $BC = b$, $CD = c$, and $DA = d$. Then, the diameter of the circumcircle of $ABCD$ is $AC = \sqrt{195}$.

By Ptolemy's Theorem, $ac + bd = \sqrt{195} \cdot BD$. Substituting into the given equation gives

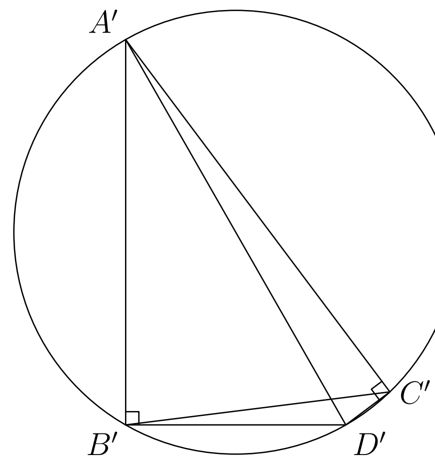
$$195 = \frac{13(ac+bd)^2}{13b^2 - 10bc + 13c^2} = \frac{13 \cdot 195 \cdot BD^2}{13b^2 - 10bc + 13c^2} = \frac{195 \cdot BD^2}{b^2 - \frac{10}{13}bc + c^2}.$$

We then have $BD^2 = b^2 + c^2 - 2bc \cdot \frac{5}{13}$. This is the Law of Cosines on $\triangle BCD$. It is easy to see that $\cos \angle BCD = \frac{5}{13}$. It follows that $\sin \angle BCD = \frac{12}{13}$. Then, by our Lemma, the area of $ABCD$ is

$$K = 2 \cdot \left(\frac{\sqrt{195}}{2} \right)^2 \cdot 1 \cdot \frac{12}{13} \cdot \sin \theta = 90 \sin \theta.$$

By the Law of Sines on $\triangle BCD$, $BD = \frac{12\sqrt{195}}{13}$. By Ptolemy's, $ac + bd = 180$.

Suppose we define the lengths of the arcs as in the proof of our lemma. Then, $\theta = \frac{w+y}{2}$.



Now consider cyclic quadrilateral $A'B'D'C'$, with $\angle A'B'D' = \angle D'C'A' = 90^\circ$ and $A'B' = a$, $B'D' = b$, $D'C' = d$, and $C'A' = c$. It follows that $\widehat{A'B'} = w$, $\widehat{B'D'} = x$, $\widehat{D'C'} = z$, and $\widehat{C'A'} = y$. Also note that $A'D' = \sqrt{195}$. Notice that

$$\sin\theta = \sin\left(\frac{w+y}{2}\right) = \sin\left(\frac{x+z}{2}\right) = \sin\angle B'A'C'.$$

By Ptolemy's, $ad + bc = \sqrt{195} \cdot B'C'$. Substituting into the given equation gives

$$195 = \frac{5(ad+bc)^2}{5a^2 - 8ac + 5c^2} = \frac{5 \cdot 195 \cdot B'C'^2}{5a^2 - 8ac + 5c^2} = \frac{195 \cdot B'C'^2}{a^2 - \frac{8}{5}ac + c^2}.$$

We then have $B'C'^2 = a^2 + c^2 - 2ac \cdot \frac{4}{5}$. Then, $\cos\angle B'A'C' = \frac{4}{5}$, and $\sin\theta = \sin\angle B'A'C' = \frac{3}{5}$. It follows that

$$K = 90\sin\theta = 54 \implies ab + cd = 108.$$

Note that $B'C' = \frac{3\sqrt{195}}{5}$ by the Law of Sines on $\triangle A'B'C'$. By Ptolemy's, $ad + bc = 117$. It follows that

$$\begin{aligned} a+b+c+d &= \sqrt{(a^2+b^2) + (c^2+d^2) + 2(ac+bd) + 2(ab+cd) + 2(ad+bc)} \\ &= \sqrt{195 + 195 + 2 \cdot 180 + 2 \cdot 108 + 2 \cdot 117} = \sqrt{1200} = 20\sqrt{3}, \end{aligned}$$

and the answer is $\lfloor 20\sqrt{3} \rfloor = 34$.