

2018 Mock AIME

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(Remastered on March 16, 2019)

Instructions

1. DO NOT BEGIN READING THE PROBLEMS UNTIL YOU HAVE STARTED YOUR TIMER.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor penalties for wrong answers.
3. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam will require the use of a calculator.
4. A combination of your AIME score and your American Mathematics Contest 12 score is *not* used to determine eligibility for participation in the nonexistent Mock USA Mathematical Olympiad (USAMO). A combination of your AIME score and your American Mathematics Contest 10 score is *not* used to determine eligibility for participation in the nonexistent USA Junior Mathematical Olympiad (USAJMO).
5. PM your answers to TheUltimate123 on AoPS.

1. The ASCII value of a digit is 48 more than the digit. For instance, the digit 0 has a value of 48, while the digit 7 has a value of 55. Let $g(n)$ be defined as the sum of the ASCII values of the digits of n for all positive integers n , when expressed in base 10. For instance, $g(10) = 97$ and $g(1234) = 202$. The sum of all positive integers n such that $g(n) = n$ is N . Find the remainder when N is divided by 1000.
2. There exist two non-intersecting circles with radii 4 and 3. Suppose the length of their common internal tangent is 21. Then, the length of their common external tangent is $\sqrt{d}$, where d is a positive integer. Find d .
3. Twelve points are chosen uniformly and at random on the circumference of a circle. The probability there exists a diameter $\overline{MN}$ such that all twelve points lie on the same side of $\overline{MN}$ is $\frac{p}{q}$ for relatively prime integers p and q . Find $p + q$.
4. For all subsets S of the set $\{1, 2, 3, \dots, 511\}$, let $p(S)$ be the product of all the elements of S , with $p(\emptyset)$ defined as 1, and $v(S)$ be the largest integer such that $2^{v(S)}$ divides $p(S)$. Over all such S , find the expected value of $v(S)$.
5. Denote by S the set of real numbers x such that $0 \leq x \leq 10000$. Let $f : S \rightarrow \mathbb{R}$ be a function such that

$$100f(xy) = f(x+y)f(x-y) + x + y$$

for all x, y such that both sides are defined. Find the remainder when $f(2018)^2$ is divided by 1000.

6. In triangle $\triangle ABC$, X lies on $\overline{AB}$ and Y lies on $\overline{AC}$ such that $\overline{BY}$ bisects $\angle ABC$ and $\overline{CX}$ bisects $\angle ACB$. $\overline{BY}$ and $\overline{CX}$ intersect at a point P . Suppose that P lies on the circumcircle of triangle $\triangle AXY$. If $AX = 15$ and $AY = 24$, find AP^2 .
7. Let $a_0 = 20$, $a_1 = 18$, and $a_n = a_{n-1} + a_{n-2}$ for all integers $n \geq 2$. There exist relatively prime positive integers m and n such that

$$\sum_{i=0}^{\infty} \frac{a_i}{3^i} = \frac{m}{n}.$$

Find $m + n$.

8. The base of a cone with radius 1 and height $\sqrt{3}$ lies on the horizontal. A plane p passes through the cone and forms an angle of 30 degrees with the horizontal. The intersection of the horizontal and p is a line that is tangent to the base. The area of the cross section formed when p passes through the cone is $N\pi$, and N^2 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
9. An ant is initially positioned at the origin. Every second, it moves 1 unit in a direction that is chosen uniformly and at random. Suppose that for all nonnegative integers n , E_n is the ant's expected distance from the origin. If K denotes the smallest positive integer such that $E_K > 2018$, find the remainder when K is divided by 1000.

10. Find the remainder when

$$\sum_{n=0}^{100} (9^n + 11^n).$$

is divided by 1000.

11. Find the smallest positive integer n such that $n^3 + 5n^2 + 2n + 2$ is divisible by 1000.
12. In triangle $\triangle ABC$, $AB = 13$, $BC = 14$, $CA = 15$, and a point P lies on $\overline{BC}$. Let Q be the foot of the perpendicular from P to $\overline{AB}$ and R be the foot of the perpendicular from P to $\overline{AC}$. Suppose I_B and I_C are the incenters of triangles $\triangle PBQ$ and $\triangle PCR$, respectively. Then the maximum possible area of $\triangle PI_B I_C$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
13. There are N ordered quadruples of positive integers (a, b, c, d) that satisfy $a + b + 2c + 3d = 89$. Find the remainder when N is divided by 1000.
14. Suppose 2019 chicks are sitting in a circle. Suddenly, each chick randomly pecks either the chick on its left or the chick on its right with equal probability. Let k be the number of chicks that were not pecked. The probability k is odd can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.
15. Let a , b , c , and d be positive real numbers such that

$$195 = a^2 + b^2 = c^2 + d^2 = \frac{13(ac + bd)^2}{13b^2 - 10bc + 13c^2} = \frac{5(ad + bc)^2}{5a^2 - 8ac + 5c^2}.$$

Find the greatest integer that does not exceed $a + b + c + d$.